STRUCTURES WITH MINIMAL TURING DEGREES AND GAMES, A RESEARCH PROPOSAL

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1. Structures whose degree spectrum contain minimal elements/turing degrees

1.1. **Background.** Richter showed in [Ric81] that if a structure A satisfies the recursive embeddability condition, then there is an isomorphic structure B such that the Turing degrees of A and B form a minimal pair. We will reformulate Richter's definition of the recursive embeddability condition that is more suitable to establish our results, and propose to use the name the computable embeddability condition that is more modern.

A consequence of this result is that if a structure satisfies the recursive embeddability condition, then it is not the case that some nontrivial (or incomputable) information is coded in every isomorphic copy of the structure. But open remains the question whether it is the case that each isomorphic copy computes some nontrivial information other than itself. That is to ask if there is an isomorphic copy the Turing degree of which is a minimal Turing degree.

We want to prove that if a structure satisfies the recursive embeddability condition and maybe some other extra condition then it has an isomorphic copy whose Turing degree is minimal.

We use the word structure in the way it is defined in any standard model theory textbook. To study the computability properties of a structure, we only consider its isomorphic copies whose domains are sets of natural numbers. Note that if S is a signature of a countable structure, we can effectively get the Gödel number of each formula in the language $S(\mathbb{N})$. We always assume that such an effective arithmetization of syntax is given when S is given, and we identify a atomic sentence in $S(\mathbb{N})$ with its Gödel number. We will identify each structure $\mathcal{A}$ of signature S whose domain is a set of natural numbers with its diagram $D(\mathcal{A})$, i.e., the set of its true atomic sentences in the language $S(\mathbb{N})$, and call a set of natural numbers A a *structure* of signature S if it is a diagram of some structure $\mathcal{A}$ of signature S. The justification to do so is that each relation and each function in $\mathcal{A}$ is computable from $A = D(\mathcal{A})$ uniformly in $\mathcal{A}$, and $A = D(\mathcal{A})$ is uniformly computable from the computable join of the domain of $\mathcal{A}$, the functions and relations of $\mathcal{A}$.

We can view strings in $2^{<\omega}$ as finite approximations of countable structures. For fixed signature S, $\sigma \in 2^{<\omega}$ is a *finite partial structure* if it codes a consistent set of atomic sentences. Terminologies in [Soaar] for finite strings in Cantor space apply to finite partial structures as well. We use $\preceq$ to denote the initial segment relation and $\prec$ the proper initial segment relation. If $\delta, \sigma, \tau \in 2^{<\omega}$, $\delta \prec \sigma$ and $\delta \prec \tau$, and σ and τ are incomparable (under $\preceq$), then we say σ and τ *split* δ.

Note that the word domain can have two meanings in our setting. It can be the domain of a structure and the domain of a function. To avoid confusion, we only use it to refer to the domain of some function. And use field(A) to mean the domain of the structure A. For a finite partial structure σ, field(σ) is the set of numbers that the sentences coded by σ have talked about.

For two structures A and B, f is an *isomorphism* from A to B if it is a bijection from field(A) to field(B) such that $\phi(n_0, ..., n_k)$ is in A if and only if $\phi(f(n_0), ..., f(n_k))$ is in B.

Given a structure A and a finite partial structure σ, we say f is an *embedding* from σ to A if f is a injection from field(σ) to field(A) such that for each atomic sentence $\phi(n_0, ..., n_k)$ that is in σ, $\phi(f(n_0), ..., f(n_k))$ is in A.

Definition 1 (The computable embeddability condition)**.** *Given a countable structure A, a finite partial structure σ, and an embedding f from σ to A, define the set of strings $A_{\sigma,f} = \{\delta : \delta$ is a finite partial structure extending σ and embeddable in A by a map which extends $f\}$.*

The structure A satisfies the recursive embeddability condition if and only if for all pairs of (σ, f) such that σ is a finite partial structure and f embedds σ into A, $A_{\sigma,f}$ is computable.

1.2. **Structures with minimal Turing degrees.** A Turing degree is minimal if there is no non-zero Truing degree strictly below it. A set A has a minimal Turing degree if and only if for each B that is Turing reducible to A, either B is computable or A is Turing reducible to B. For basics about Turing reducibility and minimal degrees, see [Soa87] and [Soaar]. Given a structure A, the degree spectrum is the set of Turing degrees that contain an isomorphic copy of A.

The following is our main theorem. The definition of the density condition will be given in Definition 7.

Theorem 2. *If an infinite structure satisfies the computable embeddability condition and the density condition then it has an isomorphic structure whose Turing degree is minimal.*

Our proof is based on Soare's presentations in [Soaar] of the existence of a minimal degree below $0''$ which was originally proved in [Spe56] and that of a minimal degree below $0'$ which was originally proved in [Sac63].

Definition 3 ([Soaar])**.** *A function tree is a partial computable function $T : 2^{<\omega} \to 2^{<\omega}$ such that for any $\alpha \in 2^{<\omega}$, if one of $T(\alpha 0)$ or $T(\alpha 1)$ is defined, then all of $T(\alpha)$, $T(\alpha 0)$ and $T(\alpha 1)$ are defined and $T(\alpha 0)$ and $T(\alpha 1)$ split $T(\alpha)$.*

It is easy to see from the definition that if T is a function tree then the domain of T is closed under $\prec$, i.e., if $T(\delta)$ is defined then $T(\sigma)$ is defined for all $\sigma \prec \delta$.

Definition 4 ([Soaar]). (1) *A string σ is on a function tree T if it is in the range of T.*
(2) *A set A is on T if every $\sigma \prec A$ is on T.*

Lemma 5. *If A is an infinite structure and $A_{\sigma,f}$ is computable, there is a total function tree $T_{\sigma,f}$ whose range is a subset of $A_{\sigma,f}$.*

Proof. Suppose $T_{\sigma,f}(\alpha)$ has been defined and is σ. Find a δ of the least length such that $\delta 0$ and $\delta 1$ are both in $A_{\sigma,f}$, and define $T_{\sigma,f}(\alpha 0) = \delta 0$ and $T_{\sigma,f}(\alpha 1) = \delta 1$. Note that such a δ is unique if exists. We know such a δ must exist because σ is in $A_{\sigma,f}$ and A is an infinite structure. □

Lemma 6. *Let T be a function tree, and S be a computable set such that every string in S that is on T can be extended into two incomparable strings in S that are on T. There is a function tree T_S such that the range of T_S is a subset of S and T_S is total if and only if T is total. Moreover T_S can be effectively obtained uniformly from T and S, and we use* Intersect(T, S) *to denote T_S where* Intersect *can be seen as a procedure.*

Proof. Let $\sigma_0 \in S$ be of the least length such that $\sigma_0 0$ and $\sigma_0 1$ are in S, and let $T_S(\epsilon) = \sigma_0$. Suppose $T_S(\alpha)$ has been defined and is σ. Find a $\sigma \succ \alpha 0$ of the least length such that both $\sigma 0$ and $\sigma 1$ is in S, and define $T_S(\alpha 0) = \sigma$. Similarly define $T_S(\alpha 1)$. □

Definition 7. *We say a structure A satisfies the density condition if for any total function tree T whose range is a subset of some $A_{\sigma,f}$, the set $\{(\tau, h) : \sigma \prec \tau, f \prec h,$ and* Intersect$(T, A_{\tau,h})$ *exists $\}$ is non-empty.*

Proof of Theorem 2. Let A be an infinite structure satisfies the computable embeddability condition, and let $T_{\sigma,f}$ has the same meaning as in Lemma 5. We will build a structure $B = \lim_{s\to\infty} \sigma_s$ and an isomerism $\lim_{s\to\infty} f_s$ from B to A by finite extensions.

Let T_{-1} be $T_{\epsilon,\emptyset}$, σ_{-1} be the empty string ϵ and f_{-1} be the empty function $\emptyset$. By Lemma 5, we know $T_{\epsilon,\emptyset}$ is a total function tree whose cover is $A_{\epsilon,\emptyset}$.

At stage $e \geq 0$, assume by induction that we have a total function tree T_{e-1}, string σ_{e-1} on T_{e-1} and f_{e-1} such that

(1) f is an embedding from σ_{e-1} into A,
(2) $0 \leq i < e$, T_{e-1} is either i-black or i-white,
(3) and the range of T_{e-1} is a subset of $A_{\sigma_{e-1}, f_{e_1}}$.

Now ask if there is e-black extension of σ_{e-1} in T_{e-1}.

Case 1: There is a $\delta \succ \sigma_{e-1}$ which is e-black. Fix such a δ. Since δ is on T_{e-1}, by induction hypothesis we know that $\delta \in A_{\sigma_{e-1}, fe-1}$. Now fix a $\sigma \succ \delta$ and $f \supset f_{e-1}$ such that the first e elements in field(A) are in the range of f and f embeds σ into A, and Intersect$(T_{e-1}, A_{\sigma,f})$

exists. Such σ and f can be found because A satisfies the density condition. Let $T_e = \text{Intersect}(T_{e-1}, A_{\sigma,f})$, σ_e be σ, and f_e be f.

Case 2: If the answer is no, then let T_e be $\text{Split}(T_{e-1}, \sigma_{e-1}, e)$, $\sigma_e = \sigma_{e_1}$, and f_e be f_{e-1}. So T_e is becomes an e-white tree. Other properties hold for e trivially because we did not extend σ_{e-1} and f_{e-1}.

The verification part goes almost the same as in the [Soaar]. The only thing to notice is that there is infinite many e such that T_e is e-black. Hence $\lim_{s\to\infty} \sigma_s$ and $\lim_{s\to\infty} f_s$ exist. □

The next step is to find natural nontrivial structures that satisfy the density condition. If we can answer this question, we are further close to getting a complete characterisation of structures that have minimal degrees in their degree spectrums.

We have shown that the degree spectrum of each well ordering has continuum many minimal Turing degrees. But that proof uses the Cooper Jump Inversion Theorem and highly relies on certain combinatorial properties of well orderings that is not shared by other classes of structures. We don't know if there are natural classes of structures that have the density property. We don't even know if there is a single non-trivial structure that satisfies the density property. Trivial structures (with empty signatures) obviously satisfy the density property. The first question we ask is the following.

Question 8. *Are there linear orderings that satisfies the density condition?*

It is natural to conjecture that the answer is yes because we already know that linear orderings A such that A is isomorphic to $\omega \cdot A$ have minimal degrees in their degree spectrum by applying the Cooper Jump Inversion Theorem. A fine analysis of the Cooper Jump Inversion Theorem might help. We will work on it but will not spend much time on it.

1.3. Axiom of Determinacy and minimal elements in degree spectrums.

We hope that Axiom of Determinacy can provide us tools for establish ing existence or non-existence of minimal elements in the degree spectrum of a structure.

First consider the following game for a given structure A which doesn't have any computable copies.

Definition 9 (Cone-avoiding Game)**.** *The first player must produce a copy of A in the end otherwise she loses. The first player controls the turns of the game, i.e., the first player determines which player to move at each round. The second player wins if he has moved only finitely often in the game, or the first player doesn't produce a copy of A, or each player produces a copy of A and the Turing degree of the copy of the second player is below that of the first player.*

It is not obvious which player has a winning strategy in this game, since the second player can not simply copy the first player's moves. If the first player has a winning strategy, it means structure A allows a kind of uniform cone-avoiding since any winning strategy for the first player provides a continuous function that maps any copy B of A to a copy that avoids the cone above B. Moreover that continuous function maps reals that are not copies of A to copies of A. If the second player has a winning strategy, it means that he can make sure his copy of A is computable from the copy of his opponent even though he has arbitrarily limited access to the moves of his opponent which he construct his copy of A.

We have another game.

Definition 10 (Second Game)**.** *The first player must produce a copy of A in the end otherwise she loses. The second player controls the turns of the game, i.e., the second player determines which player to move at each round. The second player wins if the first player doesn't produce a copy of A, or each player produces a copy of A and the Turing degree of the copy of the second player is strictly below that of the first player.*

It is easy to see that if the degree spectrum of A has a minimal element then the first player has a winning strategy. As a consequence, if we can show the second player has a winning strategy, we know that the degree spectrum of A has no minimal element.

If the second player has a winning strategy, it means there is a continuous function that maps any copy of A to another copy which has a strictly lower Turing degree. But that continuous function cannot be degree-invariant if we assume Martin's Conjecture. If the second player has a winning strategy, then there are continuum many infinite decreasing chains in the degree spectrum of A. So if we know the degree spectrum of A does not have this property then the first player has a winning strategy.

Finally we consider games for the existences of minimal elements and degrees in the degree spectrum of a structure A.

Definition 11 (Game for minimal elements)**.** *The first player is given an empty structure at the beginning and she needs to extend it to a larger finite structure in each round so that when the game finishes she has constructed an infinite structure B.*

Also in each round, the second player plays a Turing index e claiming that if $\Phi_e(B)$ is total then $C = \Phi_e(B)$ is also copy of A, and then the first player respond with playing a Turing index i claiming that $B = \Phi_i(C)$.

The second player wins the game if and only if B is not a copy of A, or for some Turing index e played by him to which the first player responded i, $C = \Phi_e(B)$ is total and a copy of A and $B \neq \Phi_i(C)$.

It is clear that if the first player has a winning strategy, then degree spectrum of A has a minimal Turing degree. Remember that we assume that A has no computable copy.

Definition 12 (Game for minimal degree)**.** *The first player is given an empty structure at the beginning and she needs to extend it to a larger finite*

TABLE 1. Game for minimal elements/degrees

	$B_0 \prec$	$B_1 \prec$	$B_1 \prec$	...
the first player	i_0	i_1	i_2	...
the second player	e_0	e_1	e_2	...

structure in each round so that when the game finishes she has constructed an infinite structure B.

Also in each round, the second player plays a Turing index e claiming that if $\Phi_e(B)$ is total then $C = \Phi_e(B)$ is also copy of A, and then the first player respond with playing a Turing index i claiming that $B = \Phi_i(C)$.

The second player wins the game if and only if B is not a copy of A, or for some Turing index e played by him to which the first player responded i, $C = \Phi_e(B)$ is a total and $B \neq \Phi_i(C)$.

It is clear that if the first player has a winning strategy, then degree spectrum of A has a minimal element.

If we assume Axiom of Determinacy, we only need to show that the second player can not have a winning strategy in order to prove the existence of minimal elements/Turing degrees in the degree spectrum of structure A. So it would be interesting if we have tools for showing that the second player cannot have a winning strategy. We believe that we can get interesting results in 2014 but will spent some time working on it.

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