EVERY SET OF NON-RANDOM STRINGS IS rK-COMPLETE

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In this paper, we follow conventions and notations in Soare's book.

We codes a set of finite strings into a real as follows. First we put the set of all finite strings in a linear order by putting $\sigma < \tau$ if $|\sigma| < |\tau|$ or $(\exists \rho) \rho\hat{}0 \preceq \sigma \wedge \rho\hat{}1 \preceq \tau$. It is easy to see that this linear ordering is effective and the empty string ϵ is the least element in this ordering. Now it is easy to identify a set S of finite strings and a real A_S which is defined by $A_S(n) = 1$ if and only if the n-th element in the above linear ordering is in S. It is easy to see that the interval $[2^i, 2^{i+1})$ in A_S contains the complete information about strings of length i in the set S.

Theorem 1. *Any set of non-random strings with respect to some optimal Turing machine is rK-complete.*

Proof. Suppose B is the set of non-random strings with respect to an optimal Turing machine U. According to Lemma 2.1 of BHLM, we only need to show that for each e there is a constant c_e such that for every l and $s < t$, if $|W_e[t] \upharpoonright l| - |W_e[s] \upharpoonright l| \geq c_e$, then $|B[t] \upharpoonright l| > |B[s] \upharpoonright l|$.

Given e, let $I_e(i) = [2^i, 2^{i+1}) \cap W_e$ and $T_e(i) = \cup_{t \leq i} I_e(i)$. For each e and p, there is a computable function $f(e,p)$ such that $W_{f(e,p)}$ has the following effective enumeration.

For each i we use a counter of size 2^{p+3} which is initially set with value 0. Whenever an element is enumerated into $T_e(i)[s]$ we increase the counter associated to i. If a counter for some i changed its value from a nonzero value to 0 then we enumerate $< i-p-2, x >$ to $W_{f(e,p)}[s]$ where x is the least string of length $i-1$ that has not been added into $W_{f(e,p)}[s]$.

It is easy to see that for each i, there are at most $2^{i+1}/2^{p+3} = 2^{i-p-2}$ request with first component $i-p-2$. Hence by Proposition 2.1.14 of Nies, there is a machine M such that $< n, x > \in W_{f(e,p)} \leftrightarrow \exists \sigma (|\sigma| = n \wedge M(\sigma) = x)$ for each n, x. In fact the index of M can be found effectively from $f(e,p)$. Thus there is a computable function g such that

$$< n, x > \in W_{f(e,p)} \rightarrow C_{\phi_{g \circ f(e,p)}}(x) \leq n.$$

Now we apply recursion theorem to $g \circ f$ and get a computable function h such that

$$\phi_{h(e)} = \phi_{g \circ f(e,h(e))}$$

and hence

$$< n, x > \in W_{f(e,h(e))} \rightarrow C_{\phi_{h(e)}}(x) \leq n \rightarrow C(x) \leq h(e) + n. \quad (1)$$

Finally observe that Lemma 2.1 holds for $A = W_e$ with $c = \max\{|T_e(h(e))| + 1, 2^{h(e)+3}\}$. Indeed, pick any l and let i be the least number such that $l \leq 2^{i+1}$. Then enumerations in $W_e \upharpoonright l$ are also enumerations in $T_e(i)$. Hence at every c enumerations in $W_e \upharpoonright l$, an enumeration of $< i-h(e)-2, x >$ where $|x| = i-1 > h(e)$ will occur in $W_{f(e,h(e))}$. By (4) $C(x) \leq h(e) + i - h(e) - 2 < |x|$ and hence x will be enumerated in $B \cap [2^{i-1}, 2^i)$, which is also an enumeration in $B \upharpoonright l$.

□

A set B is a set of d-weakly compressible strings with respect to some optimal machine U if and only if

$$(\forall \sigma \in B) K(\sigma) \leq K(|\sigma|) + |\sigma| - d \text{ where } K \text{ is } K_U. \tag{2}$$

Theorem 2. *Let B be a set of d-weakly compressible strings with respect to some optimal machine U. Then B is rK-hard, i.e., for every c.e. set X it is true that*

$$\exists c \forall n K(X \upharpoonright n \mid B \upharpoonright n) \leq c.$$

Proof. Let A be an arbitrary c.e. set and we fix an enumeration $(A[s])$ of A. Also fix a computable approximation $f[s]$ of B.

As a result of the coding theorem, there is a constant $\mathbf{c}$ such that

$$\forall b \forall n \ |\{\sigma : |\sigma| = n \wedge K(\sigma) \leq K(n) + n - d\}| \ \leq \ 2^{\mathbf{c}} 2^{n-d}.$$

See Theorem 2.2.26 of Nies. Hence we know that $\forall n \ B[s] \upharpoonright n \leq 2^{\mathbf{c}} 2^{n-d}$.

We construct a request set W and maintain a set T during the following construction.

At stage s, look for the least n such that $|B[s] \upharpoonright n| \leq 2^{\mathbf{c}} 2^{n-d}$ and $T \upharpoonright n - B[s] \upharpoonright n = \emptyset$. If such n is found, do the following.

Look for $t_{B[s] \upharpoonright n}$ in the table. If it has not been added then add it and set its value as 0. Increase the value of $t_{B[s] \upharpoonright n}$ by 1. If the value changes from p to 0 do the following. Find the largest bit $m < n$ such that $B[s](m) = 0$ and then add m to T and enumerate $< |\sigma| - d - p, \sigma >$ into W where σ is the m-th string in our fixed coding.

To see W is bounded, notice that for each n, $\log((1 - 2^{\mathbf{c}-d})n) = \log(1 - 2^{\mathbf{c}-d}) + \log(n)$. $2^{\mathbf{c}-d} n 2^{p - \log(1-2^{\mathbf{c}-d}) - \log(n)} = 2^{\mathbf{c}-d+p}/1 - 2^{\mathbf{c}-d}$. So $W_{f(p)}$ is bounded for $p \leq d - \mathbf{c} + \log(1 - 2^{\mathbf{c}-d})$.

It is easy to see that W is uniformly computable from parameter p, i.e., $W = W_{g(p)}$ for some computable function g. By the machine existence theorem for K, from $f(p)$ we effectively get a index e of Turing machine such that

$$< n, x > \in W_{f(p)} \rightarrow K_{\phi_e}(x) \leq n$$

where $e = g \circ f(p)$ for some computable function g. Apply recursion theorem to $g \circ f$ we get an index k such that $\phi_k = \phi_{g \circ f(k)}$. Hence

$$< n, x > \in W_{f(k)} \rightarrow K_{\phi_k}(x) \leq n \wedge n = |x| - k - d \rightarrow K(x) \leq |x| - d \rightarrow x \in B. \tag{3}$$

□

Theorem 3. *Any set of non-random strings with respect to some optimal prefix-free Turing machine is rK-complete.*

Proof. Suppose B is the set of non-random strings with respect to an optimal Turing machine U. According to Lemma 2.1 of BHLM, we only need to show that for c.e. set A there is a constant c such that for every l and $s < t$, if $|A[t] \upharpoonright l| - |A[s] \upharpoonright l| \geq c$, then $|B[t] \upharpoonright l| > |B[s] \upharpoonright l|$.

Given A, let $I(i) = [2^i, 2^{i+1}) \cap A$ and $T(i) = \cup_{t \leq i} I(i)$. For each p, there is a computable function $f(p)$ such that $W_{f(p)}$ has the following effective enumeration.

For each i we use a counter of size 2^{p+3} which is initially set with value 0. Whenever an element is enumerated into $T_e(i)[s]$ we increase the counter associated to i. If a counter for some i changed its value from a nonzero value to 0 then we enumerate $< i-p-d-2, x >$ to $W_{f(p)}[s]$ where x is the greatest (in the canonical order) string of length $i-1$ that has not been added into $W_{f(p)}[s]$.

It is easy to see that for each i, there are at most $2^{i+1}/2^{p+3} = 2^{i-p-2}$ request with first component $i-p-d-2$. Hence by Proposition 2.1.14 of Nies, there is a machine M such that $< n, x > \in W_{f(p)} \leftrightarrow \exists \sigma (|\sigma| = n \wedge M(\sigma) = x)$ for each n, x. In fact the index of M can be found effectively from $f(p)$. Thus there is a computable function g such that

$$< n, x > \in W_{f(p)} \rightarrow K_{\phi_{g \circ f(p)}}(x) \leq n.$$

Now we apply recursion theorem to $g \circ f$ and get a computable function h such that

$$\phi_{h(e)} = \phi_{g \circ f(e, h(e))}$$

and hence

$$< n, x > \in W_{f(e,h(e))} \rightarrow C_{\phi_{h(e)}}(x) \leq n \rightarrow C(x) \leq h(e) + n. \tag{4}$$

Finally observe that Lemma 2.1 holds for $A = W_e$ with $c = \max\{|T_e(h(e))| + 1, 2^{h(e)+3}\}$. Indeed, pick any l and let i be the least number such that $l \leq 2^{i+1}$. Then enumerations in $W_e \upharpoonright l$ are also enumerations in $T_e(i)$. Hence at every c enumerations in $W_e \upharpoonright l$, an enumeration of $< i - h(e) - 2, x >$ where $|x| = i - 1 > h(e)$ will occur in $W_{f(e,h(e))}$. By (4) $C(x) \leq h(e) + i - h(e) - 2 < |x|$ and hence x will be enumerated in $B \cap [2^{i-1}, 2^i)$, which is also an enumeration in $B \upharpoonright l$.

□