DEGREES OF NON-DETERMINACY OF INFINITE GAMES WITHOUT THE AXIOM OF CHOICE

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1. INTRODUCTION

In this paper we will study a kind of 2-player games with perfect information. Each game of this kind takes \( \omega \) steps to finish and one player wins and the other loses. We call these games infinite games for simplicity.

Blass [1] introduced the degrees of non-determinacy of infinite games and showed that they form a complete lattice. Using the axiom of choice (AC) he showed that for each cardinal \( \kappa \) such that \( 2^\omega = \kappa^\omega \), there are certain properties of degrees of non-determinacy of infinite games on \( \kappa \) (or degrees on \( \kappa \) in short). There is a connection between the degree structure of non-determinacy and linear logic established in [2].

Many of Blass’s results in [1] and [2] heavily rely on the use of AC. We are interested in exploring the structures of degrees of non-determinacy and their connections to logic in set theory systems that does not contain AC. In this paper we work with Zermelo-Fraenkel set theory (ZF) with the axiom of determinacy (AD).

It follows directly from AD that there are only two degrees on \( \omega \) and we get classical propositional logic. But what the degrees on uncountable cardinals look like are not immediately clear. In this paper we focus on a special kind of infinite games called real-coding games. We will show that if \( \kappa > \omega \) is regular under AD then the degrees on \( \kappa \) has at least four different degrees, and that the degrees on \( \Theta \) (see [6] for the definition of \( \Theta \)) have chains of order-type \( \omega_1 \) or antichains of cardinality \( \omega_1 \).

2. INFINITE GAMES

In this section, we translate our intuitions about infinite games into precise mathematics. We are primarily interested in the class of two-player win-lose (without draw) games with perfect information of length \( \omega \), and by game we always mean game in this class.

We fix two players for all games and give several names to each of them. One is player 0 who is female and the other is player 1 who is male.

If we think harder about the question “what is a game”, we will realize that the essence of a game \( A \) is nothing but its rules: at the beginning and after each move in the game, \( A \) should determine which player to move next; when infinitely many moves have been made, \( A \) should determine which player has lost. \( A \) should also provide a set \( X \) of possible moves to
the players. Each player makes his/her move by picking a possible move out of $X$. So the moves in the game $A$ occur in an $\omega$-sequence of elements of $X$. That is, when $A$ is finished after $\omega$ steps, an $s \in {}^\omega X$ is produced.

Formally, a game $A$ on a set $X \subseteq \omega$ is a function $A : \text{Seq}(X) \rightarrow 2$.

After identifying the game as its rules, it is not a surprise that the function $A$ serves a dual purpose. On finite sequences, it indicates who is to move next; on infinite sequences, it indicates who has lost the play. At move $n$, the sequence $s|n$ has already been produced and is known to both players; $s(n)$ is chosen by player $A(s|n)$. When the play $s$ is finished, i.e., $\text{length}(s) = \omega$, player $A(s)$ is the loser.

We say a prefix free set $\Sigma$ of finite sequences in $X$ is a continuation of a prefix free set $\Delta$ of finite sequences in $X$, denoted by $\Delta \gg \Sigma$, if for each $x \in \Sigma$, there is a $y \in \Delta$ such that $y < x$.

**Definition 1.** We say game $A$ on $X$ and game $B$ on $Y$ are isomorphic, denoted by $A \cong B$ if there is a function $F \subset P(^{<\omega}X) \times P(^{<\omega}Y)$ and a function $G \subset P(^{<\omega}X) \times 2$ that are defined by $\Delta^0_1$ formulas with parameters $A, B, X, Y$ and satisfies the following conditions.

1. $F$ is one to one and both $\text{dom}(F)$ and $F(\Sigma)$ contain only prefix free sets.
2. We say $\gg$ minimal sets in $\text{dom}(F)$ and $\text{ran}(F)$ are in level 0 of $F$, and for $\Sigma$ in level $n$, all its $\gg$ minimal continuations are in level $n + 1$. If $\Sigma \in \text{dom}(F)$ is in level $n$ then $F(\Sigma)$ is in level $n$; if $\Delta \in \text{ran}(F)$ is in level $n$ then $F^{-1}(\Delta)$ is in level $n$.
3. Each level is controlled by exactly one player in sense that $G(\Sigma) = G(\Delta)$ for $\Sigma$ and $\Delta$ in the same level, and for each $\Sigma \in \text{dom}(F) \cup \text{ran}(F)$, if $\Delta$ is the $\gg$ predecessor of $\Sigma$ then for $\sigma$ such that $\sigma \prec \tau$ for some $\tau \in \Sigma$ and $A(\sigma) = 1 - G(\Delta)$, both $\sigma 0$ and $\sigma 1$ have extensions in $\Sigma$.
4. For each $x \in {}^\omega X$, if for an increasing sequence $k_n$, $x|k_n$ is in $\Sigma_n$ in level $n$ of $F$ for each $n$, then the set $\{y : y|l_n$ is in $F(\Sigma_n)$ for some increasing sequence $l_n\}$ is nonempty and $B(y) = A(x)$ for every $y$ in that set; exchange $X, A$ and $Y, B$ in the above sentence.
5. If $x \in {}^\omega X$ is such that there is a $\gg$ maximal $\Sigma$ such that there is a $\sigma \in \Sigma$ and $\sigma \prec x$, then $A(x) = G(\Sigma)$.

It is easy to see that for any game $A$, we can find a game $B$ in which two players move alternatively and player 0 makes the first move such that $A \cong B$. It is also easy to see that for any game $A$ on $\omega$, we can find a game $B$ on 2 such that $A \cong B$. In general, for a game $A$ on a complicated set $X$, we can find a game $B$ isomorphic to $A$ but on a less complicated set $Y$, but with a more complicated rules. Conversely, we can make the rules simpler by making the set contain more things.

Each $a \in {}^\omega X$ is a finished play of $A$. Each $s \in \text{Fin}(X)$ is a partial play or finite play of $A$. 
Given a play \( s \) of game \( G \), let \( s|_0^G \) be the sequence of moves of player 0 in \( s \), and \( s|_1^G \) be that of player 1 in \( s \) (\( G \) is often omitted if the game is clear from the context).

Given \( s, t \in \text{Seq}(X) \), \( s \ast^G t \) is the maximal \( p \in \text{Seq}(X) \) such that \( p|_0^G \preceq s \) and \( p|_1^G \preceq t \).

3. Game Algebra

The dual of a game \( G \) is obtained by exchanging the roles of player 0 and player 1 in \( G \), and is denoted by \( \tilde{G} \). So the goal of player 0 (player 1) in \( \tilde{G} \) is to play and win as player 1 (player 0) in \( G \).

It is easy to see that player 0 (player 1) has a winning strategy in \( \tilde{G} \) if and only if player 1 (player 0) has a winning strategy in \( G \).

We will define one more operation, the tensor product \( \otimes \). It will be used in the definition of the orderings of games. The tensor product \( \otimes \mathcal{G} \) of a set of games \( \mathcal{G} \) is played as follows. At the beginning of each move of the game, player 0 chooses a game \( G \in \mathcal{G} \) and then a move is made in that chosen \( G \); this is considered one move in the game. The game is won by player 0 if and only if at least one of the \( G_i \)'s has been won by her. If \( |\mathcal{G}| = |X| \) and each \( G \in \mathcal{G} \) is a game on \( X \), then there is a game \( A \) on \( X \) that is isomorphic to \( \otimes \mathcal{G} \). In particular, for any game \( G \) and game \( H \) on \( X \), there is a game \( A \) on \( X \) such that \( A \cong G \otimes H \).

We define the dual of the tensor product \( \otimes \), the dual tensor product \( \overset{\ast}{\otimes} \). At the beginning of each move of the game, player 1 chooses a game \( G_i \) and then a move is made in that chosen \( G_i \); this is considered one move in the game. The game is won by player 1 if and only if at least one of the \( G_i \)'s has been won by him.

3.1. Reducibility. A game \( G_0 \) is reducible to a game \( G_1 \), denoted by \( G_0 \leq G_1 \) if and only player 0 has a winning strategy in the game \( G_0 \otimes \tilde{G}_1 \).

A game \( G_0 \) is equivalent to a game \( G_1 \), denoted by \( G_0 \equiv G_1 \) if and only \( G_0 \leq G_1 \) and \( G_1 \leq G_0 \).

A game \( G_0 \) is incomparable with a game \( G_1 \), denoted by \( G_0 \not\leq G_1 \) if and only \( G_0 \not\leq G_1 \) and \( G_1 \not\leq G_0 \).

Intuitively, \( G_0 \leq G_1 \) when player 0 can make sure that if she plays \( G_0 \) together with a version of \( \tilde{G}_1 \) where the roles of the two players are exchanged, and she can switch between the two games, then whenever her opponent wins \( G_1 \) as player 0 she wins \( G_0 \). In a sense player 0 know how to win \( G_0 \) if someone can show her how to win \( G_1 \) as player 0, and so \( G_0 \) is easier for her to win than \( G_1 \).

Proposition 2. If two games are incomparable, then both of them are non-determined.

Proof. Suppose \( A \) is a determined game. If \( A \) is a win for player 1 then \( B \overset{\ast}{\otimes} A \) is a win for player 1 and hence \( B \leq A \) for any \( B \). Otherwise \( A \) is a
win for player 0 and $A \otimes B$ is a win for player 0 and hence $A \leq B$ for any $B$.

Lemma 3. For any two game $A$ and $B$, player 1 has a winning strategy in $A \otimes B$ if and only if he has a winning strategy in $A$ or has a winning strategy in $B$; and similarly player 0 has a winning strategy in $A \otimes B$ if and only if she has a winning strategy in $A$ or has a winning strategy in $B$.

4. Real coding games

In the world without AC, the cardinality of $\mathbb{R}$ is not an ordinal. Instead, we can define $\Theta = \sup(\{\alpha : \text{there is a surjection } \omega \rightarrow \alpha\}$ and consider it the representative of the real numbers in the ordinals. Clearly, if AC holds, then $\Theta = (2^{\aleph_0})^+$. Without the Axiom of Choice, it is not immediately clear what $\Theta$ is. Of course $\Theta$ must be $\aleph_0$ by Theorem 1. In our setting, i.e., ZF + AD, $\Theta$ is a relatively big cardinal. In fact, Solovay proved that $\Theta$ is an $\aleph_1$-fixed point [6, Exercise 28.17].

Remember that without the Axiom of Choice, successor cardinals can be singular. So, it is not clear how many of the cardinals below $\Theta$ are regular. In the AD-situation, quite a lot is known about this. Let us start with $\aleph_1$:

Theorem 4. Assume AD. Then $\omega_1$ is regular.

Proof. Suppose $\omega_1$ is singular and $\lim_{\alpha \in \omega} \alpha_n = \omega_1$. Without loss of generality, we assume $\omega < \alpha_n$ for each $n$.

Since $|\omega \times \omega| = \omega$, we have

\[ P(\omega \times \omega) \mid P(\omega) \mid = |\omega| \cdot |\omega| = |\omega|^2. \]

Clearly $\pi : P(\omega \times \omega) \rightarrow \omega_1$ is onto.

So $\pi^{-1}(\alpha) \neq \emptyset$ for any $\alpha \in \omega_1$. Fix for each $\alpha_n$ a $R_{\alpha_n}$ such that $\pi_1(R_{\alpha_n}) = \alpha_n$. Once $R_{\alpha_n}$ is fixed, there is a unique $g_{\alpha_n} : \bigcup R_{\alpha_n} \rightarrow \alpha_n$ such that $g_{\alpha_n}$ is 1-to-1 and onto and $(k, l) \in R_{\alpha_n}$ if and only if $g_{\alpha_n}(k) \in g_{\alpha_n}(l) \in \alpha_n$. This is easy to check.

Consider the function $h : \omega \times \omega \rightarrow \omega_1$ defined by

\[ h(m, n) = \pi_1(\{(k, l) : k, l \in \omega \text{ and } (k, l), (l, m) \in R_{\alpha_n}\}) \]

which maps each $(m, n) \in \omega \times \omega$ to the order type the initial segment of $R_{\alpha_n}$ below $m$. It is easy to see $h$ is well-defined. Remember that any initial segment of any well-ordering is a well-ordering.

We show that $\text{ran}(h) = \omega_1$. $\text{ran}(h) \subset \omega_1$ is given by definition. We only need to check $\omega_1 \subset \text{ran}(h)$. Take any $\alpha \in \omega_1$. Then there is a $\alpha_j$ for some $j \in \omega$ such that $\alpha \in \alpha_j$ by our assumption that $\lim_{\alpha_n \in \omega} \alpha_n = \omega_1$. Then $\{(k, l) : k, l \in \omega \text{ and } (k, l), (l, m) \in R_{\alpha_j}\}$ is isomorphic to $\alpha$ via $g_{\alpha_j}$ because $(m, n) \in R_{\alpha_j}$ if and only if $g_{\alpha_j}(m) \in g_{\alpha_j}(n) \in \alpha_j$, and hence

\[ h(g_{\alpha_j}^{-1}(\alpha), j) = \alpha. \]

Thus we have shown $\text{ran}(h) = \omega_1$. By the fact that $\omega \times \omega$ is countable, we get that $\omega_1$ is countable. Contradiction. \qed
Steve Jackson gave a beautiful analysis of the regular cardinals below $\aleph_\epsilon_0$ in his PhD thesis [5]. We will not go into detail here, as for our present purposes, it only matters that there are uncountably many regular cardinals below $\Theta$.

**Theorem 5.** Assume AD. Then there are uncountably many regular cardinals below $\Theta$. In particular, $\omega_2$ is a regular cardinal.

*Proof.* [5]; [6, p. 388].

Among regular cardinals below $\Theta$, some are measurable. In particular, $\omega_1$ and $\omega_2$ are measurable [6, Theorem 28.2, Theorem 28.6].

### 4.1. Real-coding games

In this section we will see the interesting fact that our determinacy assumptions for games on $\omega$ implies certain games on higher cardinals are non-determined. Those non-determined games have similar style and we call them real-coding games.

If $S \subset \kappa < \Theta$ and $\pi$ is a surjection from $^{\omega}\omega$ onto $\kappa$, we consider $\pi$ as a *coding function* coding elements of $\kappa$ by reals, and define the set of $\pi$-codes for $\beta$ to be

$$C^\pi_\beta = \{ a \in {\omega}^\omega : \pi(a) = \beta \}.$$  

Each $a \in C^\pi_\beta$ is called a $\pi$-code of $\beta$. For $S \subset \kappa$, we define the real-coding game for $S$ and $\pi$ as follows.

**Definition 6.** Let $S \subset \kappa < \Theta$, $\pi : {\omega}\omega \to \kappa$ be onto. The game $G^\pi_S$ is the strict game such that

$$G^\pi_S(a) = \begin{cases} 0 & \text{if } a(0) \notin S \text{ or } \pi(a|_1) = a(0) \\ 1 & \text{otherwise} \end{cases}$$

for each $a \in {\omega}^\kappa$.

It is easy to see that the only move of $O$ that matters is her first one and her first move $\alpha$ should be an element of $S$ if she does not want to lose immediately. It is also easy to see $P$ has to play some $b \in {\omega}\omega$ if he wants to win at all. When the game is finished, $P$ wins if and only $\alpha \notin S$ or $b \in C^\pi_\alpha$.

We give some interesting facts about real-coding games in the following.

Let $\omega < \kappa < \Theta$ and $\pi : {\omega}\omega \to \kappa$ be onto. We define a function

$$FC_\pi : \kappa \to P(\text{Fin}(\omega))$$

that will give finite initial segments of $\pi$ codes of $\beta$ for each $\beta < \kappa$ as

$$FC_\pi(\beta) = \{ a|n : n < \omega \land a \in C^\pi_\beta \}$$

for each $\beta < \kappa$.

It is easy to see $FC_\pi(\beta)$ is the set of finite initial segments of $\pi$-codes of $\beta$, or finite codes of $\beta$ in short.

If $R \subset \text{Fin}(\omega)$ is such that for $\kappa$ many $\alpha$’s, we have $FC_\pi(\alpha) = R$, we say that $R$ is $FC_\pi$-maximal. The following lemma states that $FC_\pi$-maximal sets exist.
Lemma 7. There is $S \subset \kappa$ such that $|S| = \kappa$ and
\[ (\forall \alpha, \beta \in S) \text{ FC}_\pi(\alpha) = \text{ FC}_\pi(\beta). \]

Proof. It is easy to see $P(\text{Fin}(\omega))$ has cardinality continuum. We already know that there is no uncountable well-orderable subsets of $\omega$, so the range of $\text{ FC}_\pi$ must be countable. By regularity of $\kappa$, there is a some $R \in \text{ ran}(\text{ FC}_\pi)$ such that \[ \{ \beta : \text{ FC}_\pi(\beta) = R \} \] has cardinality $\kappa$. \hfill \Box

The following lemma says each $\text{ FC}_\pi$-maximal set has size of the continuum.

Lemma 8. If $R$ is $\text{ FC}_\pi$-maximal, then $R$ has size continuum.

Proof. Let $S \subset \kappa$ has cardinality $\kappa$ and such that $\text{ FC}_\pi[S] = \{ R \}$. Suppose $R$ does not have size continuum. Then $R$ is countable. $P = \{ a : (\forall n < \omega) a|n \in R \}$ is also countable. But $\pi[P] \supset S$, which has cardinality $\kappa$. Contradiction. \hfill \Box

The next lemma shows that for each $\beta \in S$, there are as many $\pi$ codes of $\beta$ as there are reals.

Lemma 9. Let $S \subset \kappa$ has cardinality $\kappa$ and such that $\text{ FC}_\pi[S] = \{ R \}$. For each $\beta \in S$, $C^\pi_\beta$ has cardinality continuum.

Proof. Consider the function
\[ (a, n) \mapsto a|n. \]
This function is a surjection from $C^\pi_\beta \times \omega$ to $R$. If $C^\pi_\beta$ is countable, then clearly $|R| = \omega$, contradicting our assumption. So $C^\pi_\beta$ must have cardinality continuum. \hfill \Box

5. DEGREES OF REAL CODING GAMES

The reason that we are interested in these real-coding games is that they are non-determined. Let $o$ be the infinite sequence of 0’s, i.e., $o : \omega \rightarrow 1$. We will need this $o$ several times in the rest of this thesis.

Theorem 10. Assume $\text{ AD}$. If $\omega < |\kappa| < \Theta$, $\pi$ is a surjection from $\omega \omega$ onto $\kappa$, $S \subset \kappa$ and $|S| > \omega$, then $G^\pi_S$ is non-determined.

Proof. Clearly $O$ does not have a winning strategy, because for each $\alpha \in S$, there is some $a \in \omega\omega$ such that $\pi(a) = \alpha$.

Suppose $P$ has a winning strategy $\sigma$.

Define the function
\[ f_\sigma(\alpha) = (\alpha \ast o) \ast \sigma|1. \]
Note that $(\alpha \ast o) \ast \sigma|1$ is the unique $t$ such that $P$ follows $\sigma$ in the play $(\alpha \ast o) \ast t$. This is a choice function for the family $\{ C^\pi_\beta : \beta \in S \}$ because $\sigma$ is winning strategy for $P$. Since $S$ is uncountable, this is a contradiction. \hfill \Box
Corollary 11 (Mycielski [7]). \( \text{AD} \) implies there is a non-determined game on \( \omega_1 \).

Proof. Fix a 1-to-1 and onto function \( f : \omega \rightarrow P(\omega \times \omega) \). Then \( \pi_0 \circ f \) is a surjection from \( \omega \) onto \( \omega_1 \).

Corollary 12. Assume \( \text{AD} \). If \( \omega < |\lambda| < \Theta \) and \( \omega < |\kappa| < \Theta \), \( \pi \) is a surjection from \( \omega \) onto \( \lambda \) and \( \rho \) is a surjection from \( \omega \) onto \( \kappa \), \( |S| > \omega \), \( S' \subset \kappa \) and \( |S'| > \omega \), then player \( O \) does not have a winning strategy in the game \( G^\pi_S \bowtie G^\rho_{S'} \).

Proof. Consider the plays in which player \( P \) never makes switches.

Another corollary is that \( \text{ZF} \) proves that all \( \text{AD}_{\omega_\alpha} \)'s are false for \( \alpha > 0 \), since \( \text{AD} \rightarrow \neg \text{AD}_{\omega_\alpha} \) and \( \neg \text{AD} \rightarrow \neg \text{AD}_{\omega_\alpha} \).

5.1. Partial Incomparability. Corollary 12 tells us player \( O \) has no winning strategy in the game \( G^\pi_S \bowtie G^\rho_{S'} \). What about player \( P \)? The following theorem says for some game of the form \( G^\pi_S \bowtie G^\rho_{S'} \), \( P \) does not have a winning strategy.

Theorem 13. Assume \( \text{AD} \). Given a cardinal number \( \lambda > \omega \) and a cardinal number \( \kappa > \lambda \), a surjection \( \pi : \omega \rightarrow \lambda \) and a surjection \( \rho : \omega \rightarrow \kappa \), the game \( G^\pi_S \bowtie G^\rho_{S'} \) is non-determined if \( |S| \) is regular and \( |S| > |S'| \). In particular, \( G^\pi_S \bowtie G^\rho_{S'} \neq G^\rho_{S'} \).

Proof. Without loss of generality, we prove the case in which \( S = \kappa \) and \( S' = \lambda \).

By Corollary 12 \( O \) does not have a winning strategy. Now let us prove \( P \) does not have a winning strategy. Suppose, towards a contradiction, that \( \sigma \) is a winning strategy for \( P \).

Notice that by definition of \( \bowtie \), the first move in the game \( G^\rho_{S'} \bowtie G^\pi_{S} \) belongs to \( O \) and she has to play an ordinal in \( G^\pi_{S} \). Now consider the set of all finished plays of \( G^\rho_{S'} \bowtie G^\pi_{S} \) in which \( P \) follows \( \sigma \) and \( O \) plays 0's in \( G^\rho_{S'} \) after her first move, and in which the the sub-game \( G^\pi_{S} \) is unfinished. Let \( \mathcal{P}_\sigma \) be the set of such plays. Formally

\[
\mathcal{P}_\sigma = \{ x \in \omega_\kappa : x = b \star \sigma \text{ for some } b \in \omega_\kappa \land (x)_{G^\kappa_{\omega}} \in \text{Fin}(\kappa) \\
\land (x)_{G^\kappa_{\omega}}[0] - (x)_{G^\kappa_{\omega}}(0) = o \}.
\]

Clearly \( |\mathcal{P}| = \kappa \). Define

\[
S_\sigma = \{ \beta \in \kappa : O \text{ plays } \beta \text{ on her first move in } G^\rho_{\omega} \text{ in some } x \in \mathcal{P} \}
\]

\[
= \{ \beta \in \kappa : (x)_{G^\kappa_{\omega}}(0) = \beta \text{ for some } x \in \mathcal{P}_\sigma \}.
\]

Lemma 14. \( S_\sigma \) is at most countable.
**Proof of Lemma 21.** Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays $w$’s of $\widehat{G}_\lambda^\pi$ (meaning $\text{Ext}(G_\lambda^\pi, \kappa)$). For each such $w$ and $\beta \in S_\sigma$, there is at most 1 play $x$ such that $x \in P_\sigma$ and $w$ is the $\widehat{G}_\lambda^\pi$ part of $x$ and $O$ plays $\beta$ in the sub-game $G_\kappa^\rho$. Formally,

$$g : (w \in \text{Fin}(\lambda), \beta \in S_\sigma) \mapsto x \in P_\sigma$$

is a partial 1-to-1 function from $\text{Fin}(\kappa) \times S_\sigma$ to $P_\sigma$. (Take two such plays $x_1$ and $x_2$. The moves of player $O$ are the same in both $x_1$ and $x_2$ and so must be the moves of $P$ since $P$ follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_\sigma$, define $T_\beta = \{(x)_{G_\lambda^\pi} | \beta \in P_\sigma \land x_{G_\lambda^\pi}(0) = \beta \}$. $T_\beta$ is a set of reals that code $\beta$, each of which is played by $P$ in $G_\kappa^\rho$ when $G_\lambda^\pi$ is unfinished. Since $g$ is 1-to-1, we know $1 \leq |T_\beta| \leq \kappa$.

By our assumption that $S$ is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_\sigma} T_\beta$. Contradiction.

Now consider all the plays of $G_\kappa^\rho \tilde{\otimes} G_\lambda^\pi$ in which $O$ plays some $\beta \in \kappa - S_\sigma$ and 0’s in $G_\kappa^\rho$. In each such play, $G_\lambda^\pi$ is finished. And it is not hard to see that the infinite sequence played by $O$ in $\widehat{G}_\lambda^\pi$ could be anything. Since $\sigma$ is a winning strategy for $P$, whenever $O$ has played a proper code in $\widehat{G}_\lambda^\pi$, $P$ must have played a proper code in $G_\kappa^\rho$.

We need to define the following auxiliary objects from the winning strategy $\sigma$: Given $\beta \in \kappa - S_\sigma$, let $r_\sigma(\beta)$ be the move in $\widehat{G}_\lambda^\pi$ that player $P$ makes according to $\sigma$ in the game $G_\lambda^\pi$ after player $O$ played $\beta$ in $G_\kappa^\rho$.

After $\beta$ and $r_\sigma(\beta)$ have been played, the winning strategy $\sigma$ gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for $\beta$, a continuous function that maps each $a \in C_\sigma^{r_\sigma(\beta)}$ to some $b \in C_\beta^\rho$.

To make it precise, let $F_\sigma^\beta = \{s : s$ is a finite play of $G_\kappa^\rho \tilde{\otimes} G_\lambda^\pi, P$ follows $\sigma$ in $s$, $(s)_{G_\kappa^\rho}|_0 \prec_\sigma \beta \ast o$ and $(s)_{G_\lambda^\pi}|_0 \in \text{Fin}(\omega)\}$ and $f_\sigma^\beta = \{(t, p) : \exists s \in F_\sigma^\beta \ t = (s)_{G_\lambda^\pi}|_0 \land p = (s)_{G_\kappa^\rho}|_0\}$. Clearly $f_\sigma^\beta \subset \text{Fin}(\omega) \times \text{Fin}(\omega)$ and $f_\sigma^\beta$ is a function since $\sigma$ is a strategy for $P$. Let $F_\sigma = \{f_\sigma^\beta : \beta \in \kappa - S_\sigma\}$. By Theorem ??,

$$|F_\sigma| \leq |P(\omega \times \omega)| = |\omega^\omega|.$$  

From $f_\sigma^\beta$ we get very naturally a continuous function $f_\sigma^\beta : C_\sigma^{r_\sigma(\beta)} \to C_\beta^\rho$ defined by

$$f_\sigma^\beta(a) = \bigcup_{n<\omega} f_\sigma^\beta(a|n).$$

From $f_\sigma^\beta$ we get very naturally a continuous function $\hat{f}_\sigma^\beta : C_\sigma^{r_\sigma(\beta)} \to C_\beta^\rho$ defined by

$$\hat{f}_\sigma^\beta(a) = \bigcup_{n<\omega} f_\sigma^\beta(a|n).$$
Because by the definition of $f^\beta_\sigma$ and the fact that $\sigma$ is a winning strategy for $P$ in $G_\kappa^\pi \overset{\sim}{\otimes} G_\lambda^\pi$, for any given $a \in C_{r_\sigma(\beta)}^\pi$, i.e., $\pi(a) = r_\sigma(\beta)$,

$$\rho\left(\bigcup_{n<\omega} f^\beta_\sigma(a|n)\right) = \beta$$

which is

$$f^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a|n) \in C_\beta.$$  

(5)

Consider the function $r_\sigma : \kappa - S_\sigma \to \lambda$. Because $|\kappa - S_\sigma| = \kappa$ and $\kappa$ is regular, there must be some $\alpha'$ in $r_\sigma(\kappa - S_\sigma)$ such that $X := \{\beta : r_\sigma(\beta) = \alpha'\}$ has cardinality $\kappa$. Take $\beta_0 \neq \beta_1$ in $X$ and consider $f^0_\sigma$ and $f^1_\sigma$. Suppose $\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'$. Then $\rho(f^0_\sigma(x)) = \beta_0$ and $\rho(f^1_\sigma(x)) = \beta_1$, which implies $f^0_\sigma \neq f^1_\sigma$ and hence $f^0_\sigma \neq f^1_\sigma$. Thus, the set $F_\sigma^X = \{f^\beta_\sigma : \beta \in X\}$ has cardinality $\kappa$. By $F_\sigma^X \subset F_\sigma$ and 3, we get a subset of $\omega^\omega$ of size $\kappa$ which contradicts AD.

**Corollary 15.** Assume AD. Given a surjection $\pi$ from $\omega^\omega$ onto $\omega_1$ and a surjection $\rho$ from $\omega^\omega$ onto $\omega_2$, $G_\omega^\pi \otimes G_\omega^\rho$ is non-determined. In particular, $G_{\omega_1}^\pi \not\leq G_{\omega_2}^\rho$.

**Corollary 16.** Assume AD. Given an infinite cardinal number $\kappa > \omega$ and a surjection $\pi : \omega^\omega \to \kappa$, if $|S|$ is regular and $|S| > |S'|$, the game $G_S^\pi \otimes G_{S'}^\pi$ is non-determined. In particular, $G_{S'}^\pi \not\leq G_S^\pi$.

**Proof.** Trivial modification of the proof of Theorem 13. \hfill \Box

**Corollary 17.** If $\kappa > \omega$ is regular under AD, $S_\kappa$ has at least four different degrees.

It would be nice if we have also shown in Theorem 13 that $G_S^\pi \not\leq G_{S'}^\pi$, because that would imply $S_\kappa$ has more than one degrees. However, we can not show that in general. Under AD, we can construct games of the form $G_\lambda^\pi \otimes G_\kappa^\pi$, with $\lambda < \kappa$ in which $P$ has a winning strategy.

Given a cardinal number $\lambda > \omega$ and a cardinal number $\kappa > \lambda$, a surjection $\pi$ from $\omega^\omega$ onto $\lambda$ and a surjection $\rho$ from $\omega^\omega$ onto $\kappa$. Define $T_\lambda^\kappa = \{\alpha \in \kappa : (\forall \beta < \alpha)(\forall \gamma < \lambda) \beta + \gamma \neq \alpha\}$. $|T_\lambda^\kappa| = \kappa$ since $\lambda^\gamma \in T_\lambda^\kappa$ for all $\gamma < \kappa$. From $\rho$ and the order-type function from $T_\lambda^\kappa$ to $\kappa$, we can get a surjection $\psi : \omega^\omega \to T_\kappa^\lambda$. Define $\chi : \omega^\omega \to \kappa$ by

$$\chi(a) = \psi(a_0) + \pi(a_1)$$

where $a_0$ is the even part of $a$ and $a_1$ is the odd part of $a$. It is easy to see that the range of $\chi$ is $\kappa$.

$P$ has the following winning strategy in $G_\lambda^\pi \otimes G_\kappa^\pi$: After $O$ plays $a$ in $G_\lambda^\pi$, switch to $G_\kappa^\pi$ and play $\lambda + a$. Keep switching between two games while copying the odd part of $O$’s play in $G_\kappa^\pi$ into your $(P$’s) own code played in $G_\lambda^\pi$. 


We apply the partition property of $\omega_1$ to certain family of $\omega_1$ many games to get results about comparability among these games.

**Lemma 18.** Let $\pi: \omega_1 \to \omega_1$ be onto. Let $\{S_\alpha\}_{\alpha<\omega_1}$ be a partition of $\omega_1$ such that $|S_\alpha| = \omega_1$ for all $\alpha < \omega_1$. There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $G_{S_\alpha}^\pi \parallel G_{S_\beta}^\pi$ for all $\alpha, \beta \in I$ or 2. $G_{S_\alpha}^\pi \leq G_{S_\beta}^\pi$ or $G_{S_\beta}^\pi \leq G_{S_\alpha}^\pi$ for all $\alpha, \beta \in I$.

**Proof.** Easy application of $\omega_1 \to (\omega_1)^2$. □

The same argument can be applied to the first $\omega_1$ regular cardinals. Let $\kappa_\alpha$ be the $\alpha$-th uncountable regular cardinal under AD. By Theorem 5 these are all less than $\Theta$, so there is a surjections $\rho_\alpha$ from $\omega$ onto each $\kappa_\alpha$. Moreover, these $\rho_\alpha$’s can be precisely defined without using any choice; see [6, p. 397–398]. Let $H_\alpha$ be the game $G_\rho_\alpha^\kappa_\alpha$. Let $H$ be the set of all these $H_\alpha$’s.

**Theorem 19.** There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $H_\alpha \parallel H_\beta$ for all $\alpha, \beta \in I$ or 2. $H_\alpha > H_\beta$ for all $\alpha, \beta \in I$. That is, there is an uncountable strictly increasing sequence of games in $H$, or there is an uncountable set of incomparable games in $H$.

**Proof.** For any $\alpha < \beta$, $H_\alpha \not\leq H_\beta$ by Theorem 13 and hence either $H_\alpha > H_\beta$ or $H_\alpha || H_\beta$.

Define the following colouring function:

$$c(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } \alpha < \beta \rightarrow H_\alpha > H_\beta, \\ 0 & \text{otherwise}. \end{cases}$$

The partition property gives us a homogeneous set $T$ for $c$ of cardinality $\omega_1$, and this is either an uncountable strictly increasing sequence of games (if it is homogeneous for 1) or an uncountable set of incomparable games (if it is homogeneous for 0). □

**Corollary 20.** The degree universe $S_\Theta$ has chains of order-type $\omega_1$ or antichains of cardinality $\omega_1$.

### 6. THE NEUCL ASSUMPTION

Consider the following question: given a surjection $\pi: \omega \to \kappa$ where $\kappa$ is an infinite regular cardinal $> \omega$, what $S$ and $S'$ satisfy that $|S'| = |S| = \kappa$ and $P$ has no winning strategy in $G_S^\pi \otimes G_{S'}^\pi$?

Let $\kappa > \omega$ be a regular cardinal. Consider the game $G_{S_0}^\pi \otimes G_{S_1}^\pi$ where $S_0, S_1 \subset \kappa$ and $|S_0| = |S_1| = \kappa$. By Corollary 12 $O$ does not have a winning strategy. Now let us suppose that $\sigma$ is a winning strategy for $P$. We do the same analysis as in the proof of Theorem 13.

Notice that by definition of $\otimes$, the first move in the game $G_{S_0}^\pi \otimes G_{S_1}^\pi$ belongs to $O$ and she has to play an ordinal in $G_{S_0}^\pi$. Now consider the set of all finished plays of $G_{S_0}^\pi \otimes G_{S_1}^\pi$ in which $P$ follows $\sigma$ and $O$ plays 0’s in
Let $S_0$ be the set of such plays. Formally

$$P_{\sigma} = \{ x \in \omega : x = b \star \sigma \text{ for some } b \in \omega \wedge (x)G_{S_1}^{\pi} \in \text{Fin}(\kappa)$$

$$\wedge (x)G_{S_0}^{\pi}[0 - (x)G_{S_0}^{\pi}(0) = o] \}.$$

Clearly $|P| = \kappa$. Define

$$S_{\sigma} = \{ \beta \in S_0 : O \text{ plays } \beta \text{ on her first move in } G_{S_0}^{\pi} \text{ in some } x \in P \}$$

$$= \{ \beta \in S_0 : (x)G_{S_0}^{\pi}(0) = \beta \text{ for some } x \in P_{\sigma} \}.$$

**Lemma 21.** The set $S_{\sigma}$ is at most countable.

**Proof of Lemma 21.** Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays $w$’s of $G_{S_1}^{\pi}$ (meaning $\text{Ext}(G_{S_1}^{\pi}, \kappa)$). For each such $w$ and $\beta \in S_{\sigma}$, there is at most 1 play $x$ such that $x \in P_{\sigma}$, $w$ is the $G_{S_1}^{\pi}$ part of $x$ and $O$ plays $\beta$ in the sub-game $G_{S_0}^{\pi}$.

$$g : (w \in \text{Fin}(\lambda), \beta \in S_{\sigma}) \mapsto x \in P_{\sigma} \text{ such that and } (x)G_{S_1}^{\pi} = w \text{ and } xG_{S_0}^{\pi}(0) = \beta.$$

is a partial 1-to-1 function from $\text{Fin}(\kappa) \times S_{\sigma}$ to $P_{\sigma}$. (Take two such plays $x_1$ and $x_2$. The moves of player $O$ are the same in both $x_1$ and $x_2$ and so must be the moves of $P$ since $P$ follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_{\sigma}$, define $T_\beta = \{(x)G_{S_0}^{\pi}[0 : x \in P_{\sigma} \wedge xG_{S_0}^{\pi}(0) = \beta\}$. $T_\beta$ is a set of reals that code $\beta$, each of which is played by $P$ in $G_{S_0}^{\pi}$ when $G_{S_1}^{\pi}$ is unfinished. Since $g$ is 1-to-1, we know $1 \leq |T_\beta| \leq \kappa$.

By our assumption that $S$ is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_{\sigma}} T_\beta$. Contradiction.

Now consider all the plays of $G_{S_0}^{\pi} \otimes G_{S_1}^{\pi}$ in which $O$ plays some $\beta \in S_0 - S_{\sigma}$ and $0$’s in $G_{S_0}^{\pi}$. In each such play, $G_{S_1}^{\pi}$ is finished. And it is not hard to see that the infinite sequence played by $O$ in $G_{S_1}^{\pi}$ could be anything. Since $\sigma$ is a winning strategy for $P$, whenever $O$ has played a proper code in $G_{S_1}^{\pi}$, $P$ must have played a proper code in $G_{S_0}^{\pi}$.

We need to define the following auxiliary objects from the winning strategy $\sigma$: Given $\beta \in S_0 - S_{\sigma}$, let $r_\sigma(\beta)$ be the move in $G_{S_1}^{\pi}$ that player $P$ makes according to $\sigma$ in the game $G_{S_1}^{\pi}$ after player $O$ played $\beta$ in $G_{S_0}^{\pi}$.

After $\beta$ and $r_\sigma(\beta)$ have been played, the winning strategy sigma gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for $\beta$, a continuous function that maps each $a \in C_{r_\sigma(\beta)}$ to some $b \in C_\beta$.

To make it precise, let $F_\sigma^\beta = \{ s : s \text{ is a finite play of } G_{S_0}^{\pi} \otimes G_{S_1}^{\pi}, P \text{ follows } \sigma \text{ in } s, (s)G_{S_0}^{\pi}[0 \prec \beta * o \text{ and } (s)G_{S_1}^{\pi}[0 \in \text{Fin}(\omega)] \} \text{ and } F_\sigma^\beta = \{(t, p) : (s) \in F_\sigma^\beta \}$.
\(F^\beta_\sigma\) \(t = (s)G^\beta_\sigma|_0 \land p = (s)G^\beta_\sigma|_1\). Clearly \(f^\beta_\sigma \subseteq \text{Fin}(\omega) \times \text{Fin}(\omega)\) and \(f^\beta_\sigma\) is a function since \(\sigma\) is a strategy for \(P\). Let \(F_\sigma = \{f^\beta_\sigma : \beta \in S_0 - S_\sigma\}\). As before,
\[
|F_\sigma| \leq |P(\omega \times \omega)| = |\omega^\omega|.
\]
From \(f^\beta_\sigma\) we get very naturally a continuous function \(\hat{f}^\beta_\sigma : C^\pi_{\sigma_\beta} \rightarrow C^\pi_\beta\) defined by
\[
(7) \quad \hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a|n).
\]
Because by the definition of \(f^\beta_\sigma\) and the fact that \(\sigma\) is a winning strategy for \(P\) in \(G^\pi_{S_0} \overset{\infty}{\approx} G^\pi_{S_1}\), for any given \(a \in C^\pi_{\sigma_\beta}\), i.e., \(\pi(a) = r_\sigma(\beta)\),
\[
\pi\big(\bigcup_{n<\omega} f^\beta_\sigma(a|n)\big) = \beta
\]
which is
\[
(8) \quad \hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a|n) \in C^\pi_\beta.
\]
Consider the function \(r_\sigma : S_0 - S_\sigma \rightarrow S_1\).

**Lemma 22.** \(|\text{ran}(r_\sigma)| = \kappa\).

**Proof.** Because \(|S_0 - S_\sigma| = \kappa\) and \(\kappa\) is regular, there must be some \(\alpha' \in \text{ran}(r_\sigma)\) such that \(X := \{\beta : r_\sigma(\beta) = \alpha'\}\) has cardinality \(\kappa\). Take \(\beta_0 \neq \beta_1\) in \(X\) and consider \(f^{\beta_0}_\sigma\) and \(f^{\beta_1}_\sigma\). Suppose \(\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'\). Then \(\pi(f^{\beta_0}_\sigma(x)) = \beta_0\) and \(\pi(f^{\beta_1}_\sigma(x)) = \beta_1\), which implies \(f^{\beta_0}_\sigma \neq f^{\beta_1}_\sigma\) and hence \(\hat{f}^{\beta_0}_\sigma \neq \hat{f}^{\beta_1}_\sigma\). Thus, the set \(F_\sigma = \{f^\beta_\sigma : \beta \in X\}\) has cardinality \(\kappa\). By \(F_\sigma \subset F_\sigma\) and \(6\), we get a subset of \(\omega^\omega\) of size \(\kappa\) which contradicts AD. \(\square\)

Define \(g_\sigma : \text{ran}(r_\sigma) \rightarrow S_0 - S_\sigma\) by
\[
g_\sigma(\alpha) = \text{the least } \beta \in S_0 - S \text{ such that } r_\sigma(\beta) = \alpha.
\]
Clearly \(g_\sigma\) is 1-1 to-1 and has range of cardinality \(\kappa\). It is easy to see \(r_\sigma\) is 1-to-1 on \(\text{ran}(g_\sigma)\). Since \(F_\sigma\) is countable and \(\kappa\) is regular, we know there is \(S_0 \cap \text{ran}(g_\sigma)\) and \(f \in F_\sigma\) such that \(|S_0| = \kappa\) and \((\forall \beta, \beta' \in S_0^*) \ f^\beta_\sigma = f^\beta_\sigma\). Thus we get a continuous function \(\hat{f} : \bigcup_{\beta \in S_0^*} C^\pi_{r_\sigma(\beta)} \rightarrow \bigcup_{\beta \in S_0^*} C^\pi_\beta\). \(\hat{f}\) is a universal reduction function in the sense that \(\hat{f}|C^\pi_{r_\sigma(\beta)} = f^\beta_\sigma\), or equivalently
\[
(\forall \beta \in S_0^*) \ f[C^\pi_{r_\sigma(\beta)}] \subset C^\pi_\beta.
\]

**Definition 23.** Let \(\pi : \omega^\omega \rightarrow \kappa\) be a surjection, \(S \subset \kappa\) and \(g : S \rightarrow \kappa\) be 1-to-1. A continuous function \(f\) that is (partially) defined on \(\omega^\omega\) is a \(\pi\)-universal continuous lifting for \(S\) and \(g\) iff
\[
(\forall \beta \in S)\ f[C^\pi_\beta] \subset C^\pi_{g(\beta)}.
\]
It is easy to see the above $\hat{f}$ is $\pi$-universal for $r_\sigma[S_0^*]$ and $g_\sigma$.

From the above discussion, we know that if such a universal continuous lifting cannot exists, then we have two games that are incomparable. We now prove this more precisely.

Let $\kappa > \omega$ be a regular cardinal and $\pi : \omega \to \kappa$ be a surjection. We say the NEUCL Assumption (NEUCL for “non-existence of universal continuous lifting”) holds for the pair $(\kappa, \pi)$ if and only if there is no $\pi$-universal continuous lifting $f$ for any $S \subset \kappa$ with $|S| = \kappa$ and any 1-to-1 $g : S \to \kappa$ such that $g \cap \text{id} = \emptyset$.

Lemma 24. Assume AD. Let $\kappa > \omega$ be a regular cardinal and $\pi : \omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for $(\kappa, \pi)$, then for any $S_0$ and $S_1$ such that $|S_0| = |S_1| = \kappa$ and $|S_0 \cap S_1| < \kappa$, it is true that $G_\pi S_0 \parallel G_\pi S_1$.

Proof. By the above analysis, $P$ does not have a winning strategy in either $G_\pi S_0 \otimes G_\pi S_1$ or $G_\pi S_1 \otimes G_\pi S_0$; by Corollary 12, $O$ does not have a winning strategy in either $G_\pi S_0 \otimes G_\pi S_1$ or $G_\pi S_1 \otimes G_\pi S_0$. □

It is not hard to see that if the premiss of Lemma 24 holds, we get a large family of pairwise incomparable games. The following theorem says that if the NEUCL Assumption holds for some $(\kappa, \pi)$, then $S_\kappa$ has an antichain of size $\kappa$.

Theorem 25. Assume AD. Let $\kappa > \omega$ be a regular cardinal and $\pi : \omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for $(\kappa, \pi)$, then there is a family of games on $\kappa$ $\{G_\alpha : \alpha \in \kappa\}$ such that $G_\alpha \parallel G_\beta$ for any $\alpha \neq \beta$.

Proof. Use the canonical partition of $\kappa$ and get $S_\alpha = \kappa|\alpha$ for each $\alpha \in \kappa$. Apply Lemma 24 to $S_\alpha$ and $S_\beta$ for any $\alpha \neq \beta$. □

Corollary 26. Let $\kappa > \omega$ be a regular cardinal under AD. If there is a onto function $\pi : \omega \to \kappa$ such that the NEUCL Assumption holds for $(\kappa, \pi)$, then $S_\kappa$ has antichains of size $\kappa$.

The NEUCL Assumption cannot hold for each pair $(\kappa, \pi)$. The following is a counterexample.

Recall that $\Gamma[\omega \times \omega : \omega \times \omega \to \omega$ is 1-to-1 and onto. Define $R : \omega^2 \to P(\omega \times \omega)$ by

$$(m, n) \in R(a) \text{ iff } f(\Gamma(m, n)) = 1$$

for each $a \in \omega^2$. Clearly $R$ is 1-to-1 and onto. A nice property of $R$ that will be useful is that $R(a)[n \times n]$ can be read from the first $\Gamma(0, n)$ bits of $a$. It is so by our choice of $\Gamma$, which was defined earlier.

Define $\pi : \omega \to \omega_1$ by

$$\pi(a) = \begin{cases} 
\pi_0 \circ R(a) & \text{if } a \in \omega^2, \\
0 & \text{otherwise}.
\end{cases}$$
Recall that $\pi_0$ is the canonical projection: $P(\omega \times \omega) \to \omega_1$. So $\pi$ is a well-defined projection: $\omega \to \omega_1$.

Let $S_1 = \{\omega^\alpha : \alpha < \omega_1\}$ and $S_0 = \{\omega^\alpha \cdot 2 : \alpha < \omega_1\}$. Clearly $|S_0| = |S_1| = \kappa$ and $S_0 \cap S_1 = \emptyset$. We show that $P$ has the following winning strategy $\sigma$ in $G^n_{S_0} \otimes G^n_{S_1}$.

By definition of $\otimes$, the game starts with $O$ play a $\alpha \in S_0$ in the sub-game $G^n_{S_0}$. After $O$’s starting move, $P$ play the unique $\beta$ such that $\beta \cdot 2 = \alpha$ in the sub-game $G^n_{S_1}$. Then $O$ must begin to play a code of $\beta$, if she wants to win at all. Whenever $O$ has played $\Gamma(0, n + 1)$ bits of her code $b$, interrupt her and extend your ($P$’s) code $a$ in $G^n_{S_0}$ to the first $\Gamma(0, 2n)$ bits so that

1. for all $m, k < n$, if $b(\Gamma(m, k)) = 1$ then $a(\Gamma(2m + 1, 2k + 1)) = 1$
2. for all $m, m', k, k' < n$, if $b(\Gamma(m, k)) = 1$ and $b(\Gamma(m', k')) = 1$, then $(\forall x, y \in \{m, m', k, k'\}) a(\Gamma(2x, 2y + 1)) = 1$
3. $a(k) = 0$ for all $k < 2n$ not covered by the above clauses.

After extending $a$ to the first $2n$ bits, switch to $G^n_{S_1}$ and let $O$ continue to play her code $b$.

It is easy to see this is a well-defined strategy. Note that if $O$ starts to play anything other than 0-1 bits at any point, you just need to sit and watch and you win the game when it is finished.

It is easy to see for each $n < \omega$, if $b|n$ codes a well-ordering of order-type $m$, $a|2n$ codes a well-ordering of order-type $m \cdot 2$, and when the game is finished, if $b$ codes a well-ordering of order-type $\alpha \in S_0$, $a$ codes $\alpha \cdot 2$. So the strategy is a winning one for $P$.

From the definition of $P$’s strategy $\sigma$, $F_\sigma^\beta = F_\sigma^{\beta'}$ and hence $f_\sigma^\beta = f_\sigma^{\beta'} = f_\sigma$ for each $\beta, \beta' \in S_0$. $f_\sigma$ is $\pi$-universal for $S_0$ and $g : \beta \mapsto \beta \cdot 2$, i.e.,

$$(\forall \beta \in S_0) \ f_\sigma[C_\beta^\omega] \subseteq C_\beta^{\omega_2}.$$ 

We have seen that the NEUCL Assumption cannot be a general theorem for all pair $(\kappa, \pi)$ where $\kappa$ is regular under $\text{AD}$ and $\pi : \omega \omega \to \kappa$ is onto. It is not easy to see what $(\kappa, \pi)$ can validate the NEUCL Assumption. Now let us consider given $(\kappa, \pi)$ what $S \subset \kappa$ and $g : S \to \kappa$ cannot falsify the NEUCL Assumption.

First, we consider the case in which $S \subset \kappa = \omega_1$ and $g : S \to S$ is 1-to-1 and onto, and

$$(\forall \alpha \in S) \ g(\alpha) \neq \alpha.$$

**Lemma 27.** There is a set $T \subset S$ of cardinality $\kappa$ such that $g[T] \cap T = \emptyset$

**Proof.** It is well known that $\text{AD}$ implies $\omega_1 \to (\omega_1)^2_2$.

Define a colouring function $c : [\omega]^2 \to 2$ by

$$c((\alpha, \beta)) = \begin{cases} 1 & \text{if } g(\alpha) = \beta \text{ or } g(\beta) = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$
By weakly compactness, there is a homogenous set $T \subset S$ of cardinality $\kappa$. We claim $c[T] = \{0\}$.

Suppose otherwise, $c[T] = \{1\}$. Let $\alpha, \beta, \gamma$ and $\delta$ are four different elements of $T$. Without loss of generality, we can assume that $g(\alpha) = \beta$, which implies $g(\alpha) \neq \gamma, \delta$. But since $c(\{\alpha, \gamma\}) = c(\{\alpha, \delta\}) = 1$, it has to be the case $g(\gamma) = g(\delta) = \alpha$. But that means that $g(\gamma) \neq \delta$ and $g(\delta) \neq \gamma$ and hence $c(\{\gamma, \delta\}) = 0$. Contradiction.

By the definition of $c$, $g(\alpha) \neq \beta$ for all $\alpha, \beta \in T$ and hence $g[T] \cap T = \emptyset$. □

Clearly if $f$ is a $\pi$-universal continuous lifting for $S$ and $g$, it is also a $\pi$-universal continuous lifting for $T$ and $g$. So if we can show that for any $S \subset \omega_1$ and any $g$ such that $|S| = \omega_1$ and $g[S] \cap S = \emptyset$, there is no $\pi$-universal continuous lifting, then we also have that for any $S \subset \omega_1$ and any 1-to-1 and onto $g : S \to S$ such that $(\forall \alpha \in S) g(\alpha) \neq \alpha$, there is no $\pi$-universal continuous lifting.

REFERENCES