DEGREES OF NON-DETERMINACY OF INFINITE GAMES WITHOUT THE AXIOM OF CHOICE

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1. INTRODUCTION

In this paper we will study a kind of 2-player games with perfect information. Each game of this kind takes ω steps to finish and one player wins and the other loses. We call these games *infinite games* for simplicity.

Blass [1] introduced the *degrees of non-determinacy* of infinite games and showed that they form a complete lattice. Using the axiom of choice (AC) he showed that for each cardinal κ such that $2^\kappa = \kappa^\omega$, there are certain properties of degrees of non-determinacy of infinite games on κ (or degrees on κ in short). There is a connection between the degree structure of non-determinacy and linear logic established in [2].

Many of Blass's results in [1] and [2] heavily rely on the use of AC. We are interested in exploring the structures of degrees of non-determinacy and their connections to logic in set theory systems that does not contain AC. In this paper we work with Zermelo-Fraenkel set theory (ZF) with the axiom of determinacy (AD).

It follows directly from AD that there are only two degrees on ω and we get classical propositional logic. But what the degrees on uncountable cardinals look like are not immediately clear. In this paper we focus on a special kind of infinite games called *real-coding games*. We will show that if $\kappa > \omega$ is regular under AD then the degrees on κ has at least four different degrees, and that the degrees on Θ (see [6] for the definition of Θ) have chains of order-type ω_1 or antichains of cardinality ω_1.

2. INFINITE GAMES

In this section, we translate our intuitions about infinite games into precise mathematics. We are primally interested in the class of two-player win-lose (without draw) games with perfect information of length ω, and by game we always mean game in this class.

We fix two players for all games and give several names to each of them. One is player 0 who is female and the other is player 1 who is male.

If we think harder about the question "what is a game", we will realize that the essence of a game A is nothing but its rules: at the beginning and after each move in the game, A should determine which player to move next; when infinitely many moves have been made, A should determine which player has lost. A should also provide a set X of possible moves to the players. Each player makes his/her move by picking a possible move out of X. So the moves in the game A occur in an ω-sequence of elements of X. That is, when A is finished after ω steps, an $s \in {}^{\omega}X$ is produced.

Formally, a **game** A on a set $X \subseteq \omega$ is a function $A : \mathrm{Seq}(X) \to 2$.

After identifying the game as its rules, it is not a surprise that the function A serves a dual purpose. On finite sequences, it indicates who is to move next; on infinite sequences, it indicates who has lost the play. At move n, the sequence $s\restriction n$ has already been produced and is known to both players; $s(n)$ is chosen by player $A(s\restriction n)$. When the play s is finished, i.e., $\mathrm{length}(s) = \omega$, player $A(s)$ is the loser.

We say a prefix free set Σ of finite sequences in X is a *continuation* of a prefix free set Δ of finite sequences in X, denoted by $\Delta \gg \Sigma$, if for each $x \in \Sigma$, there is a $y \in \Delta$ such that $y \prec x$.

Definition 1. *We say game A on X and game B on Y are isomorphic, denoted by $A \cong B$ if there is a function $F \subset P({}^{<\omega}X) \times P({}^{<\omega}Y)$ and a function $G \subset P({}^{<\omega}X) \times 2$ that are defined by Δ^0_1 formulas with parameters A, B, X, Y and satisfies the following conditions.*

(1) F is one to one and both $\mathrm{dom}(F)$ and $F(\Sigma)$ contain only prefix free sets.
(2) We say $\gg$ minimal sets in $\mathrm{dom}(F)$ and $\mathrm{ran}(F)$ are in level 0 of F, and for Σ in level n, all its $\gg$ minimal continuations are in level $n+1$. If $\Sigma \in \mathrm{dom}(F)$ is in level n then $F(\Sigma)$ is in level n; if $\Delta \in \mathrm{ran}(F)$ is in level n then $F^{-1}(\Delta)$ is in level n.
(3) Each level is controlled by exactly one player in sense that $G(\Sigma) = G(\Delta)$ for Σ and Δ in the same level, and for each $\Sigma \in \mathrm{dom}(F) \cup \mathrm{ran}(F)$, if Δ is the $\gg$ predecessor of Σ then for σ such that $\sigma \prec \tau$ for some $\tau \in \Sigma$ and $A(\sigma) = 1 - G(\Delta)$, both $\sigma 0$ and $\sigma 1$ have extensions in Σ.
(4) For each $x \in {}^{\omega}X$, if for an increasing sequence k_n, $x\restriction k_n$ is in Σ_n in level n of F for each n, then the set $\{y : y\restriction l_n$ is in $F(\Sigma_n)$ for some increasing sequence $l_n\}$ is nonempty and $B(y) = A(x)$ for every y in that set; exchange X, A and Y, B in the above sentence.
(5) If $x \in {}^{\omega}X$ is such that there is a $\gg$ maximal Σ such that there is a $\sigma \in \Sigma$ and $\sigma \prec x$, then $A(x) = G(\Sigma)$.

It is easy to see that for any game A, we can find a game B in which two players move alternatively and player 0 makes the first move such that $A \cong B$. It is also easy to see that for any game A on ω, we can find a game B on 2 such that $A \cong B$. In general, for a game A on a complicated set X, we can find a game B isomorphic to A but on a less complicated set Y, but with a more complicated rules. Conversely, we can makes the rules simpler by making the set contain more things.

Each $a \in {}^{\omega}X$ is a **finished play** of A. Each $s \in \mathrm{Fin}(X)$ is a **partial play** or **finite play** of A.

Given a play s of game G, let $s|_0^G$ be the sequence of moves of player 0 in s, and $s|_1^G$ be that of player 1 in s (G is often omitted if the game is clear from the context).

Given $s, t \in \mathrm{Seq}(X)$, $s *^G t$ is the maximal $p \in \mathrm{Seq}(X)$ such that $p|_0^G \preceq s$ and $p|_1^G \preceq t$.

3. GAME ALGEBRA

The *dual* of a game G is obtained by exchanging the roles of player 0 and player 1 in G, and is denoted by $\widetilde{G}$. So the goal of player 0 (player 1) in $\widetilde{G}$ is to play and win as player 1 (player 0) in G.

It is easy to see that player 0 (player 1) has a winning strategy in $\widetilde{G}$ if and only if player 1 (player 0) has a winning strategy in G.

We will define one more operation, the *tensor product* $\bigotimes$. It will be used in the definition of the orderings of games. The tensor product $\bigotimes \mathcal{G}$ of a set of games $\mathcal{G}$ is played as follows. At the beginning of each move of the game, player 0 chooses a game $G \in \mathcal{G}$ and then a move is made in that chosen G; this is considered one move in the game. The game is won by player 0 if and only if at least one of the G_i's has been won by her. If $|\mathcal{G}| = |X|$ and each $G \in \mathcal{G}$ is a game on X, then there is a game A on X that is isomorphic to $\bigotimes \mathcal{G}$. In particular, for any game G and game H on X, there is a game A on X such that $A \cong G \otimes H$.

We define the dual of the tensor product $\bigotimes$, the *dual tensor product* $\widetilde{\bigotimes}$. At the beginning of each move of the game, player 1 chooses a game G_i and then a move is made in that chosen G_i; this is considered one move in the game. The game is won by player 1 if and only if at least one of the G_i's has been won by him.

3.1. Reducibility.

A game G_0 is reducible to a game G_1, denoted by $G_0 \leq G_1$ if and only player 0 has a winning strategy in the game $G_0 \otimes \widetilde{G}_1$.

A game G_0 is equivalent to a game G_1, denoted by $G_0 \equiv G_1$ if and only $G_0 \leq G_1$ and $G_1 \leq G_0$.

A game G_0 is incomparable with a game G_1, denoted by $G_0 \| G_1$ if and only $G_0 \not\leq G_1$ and $G_1 \not\leq G_0$.

Intuitively, $G_0 \leq G_1$ when player 0 can make sure that if she plays G_0 together with a version of G_1 where the roles of the two players are exchanged, and she can switch between the two games, then whenever her opponent wins G_1 as player 0 she wins G_0. In a sense player 0 know how to win G_0 if someone can show her how to win G_1 as player 0, and so G_0 is easier for her to win than G_1.

Proposition 2. *If two games are incomparable, then both of them are non-determined.*

Proof. Suppose A is a determined game. If A is a win for player 1 then $\widetilde{B} \tilde{\otimes} A$ is a win for player 1 and hence $B \leq A$ for any B. Otherwise A is a win for player 0 and $A \otimes \widetilde{B}$ is a win for player 0 and hence $A \leq B$ for any B. □

Lemma 3. *For any two game A and B, player 1 has a winning strategy in $A \otimes B$ if and only if he has a winning strategy in A or has a winning strategy in B; and similarly player 0 has a winning strategy in $A \tilde{\otimes} B$ if and only if she has a winning strategy in A or has a winning strategy in B.*

4. Real coding games

In the world without **AC**, the cardinality of $\mathbb{R}$ is not an ordinal. Instead, we can define $\Theta = \sup(\{\alpha :$ there is a surjection: ${}^{\omega}\omega \to \alpha\}$

and consider it the representative of the real numbers in the ordinals. Clearly, if **AC** holds, then $\Theta = (2^{\aleph_0})^+$. Without the Axiom of Choice, it is not immediately clear what Θ is. Of course Θ must be $> \omega_1$ by Theorem **??**. In our setting, i.e., $\mathbf{ZF} + \mathbf{AD}$, Θ is a relatively big cardinal. In fact, Solovay proved that Θ is an $\aleph$-fixed point [6, Exercise 28.17].

Remember that without the Axiom of Choice, successor cardinals can be singular. So, it is not clear how many of the cardinals below Θ are regular. In the **AD**-situation, quite a lot is known about this. Let us start with $\aleph_1$:

Theorem 4. *Assume* **AD**. *Then ω_1 is regular.*

Proof. Suppose ω_1 is singular and $lim_{n\in\omega}\alpha_n = \omega_1$. Without loss of generality, we assume $\omega < \alpha_n$ for each n.

Since $|\omega \times \omega| = \omega$, we have

$$|P(\omega \times \omega)| = |P(\omega)| = |{}^{\omega}\omega|. \tag{1}$$

Clearly $\pi_1 : P(\omega \times \omega) \to \omega_1$ is onto.

So $\pi_1^{-1}(\alpha) \neq \emptyset$ for any $\alpha \in \omega_1$. Fix for each α_n a R_{α_n} such that $\pi_1(R_{\alpha_n}) = \alpha_n$. Once R_{α_n} is fixed, there is a unique $g_{\alpha_n} : \bigcup\bigcup R_{\alpha_n} \to \alpha_n$ such that g_{α_n} is 1-to-1 and onto and $(k,l) \in R_{\alpha_n}$ if and only if $g_{\alpha_n}(k) \in g_{\alpha_n}(l) \in \alpha_n$. This is easy to check.

Consider the function $h : \omega \times \omega \to \omega_1$ defined by

$$h(m,n) = \pi_1(\{(k,l) : k,l \in \omega \text{ and } (k,l),(l,m) \in R_{\alpha_n}\})$$

which maps each $(m,n) \in \omega \times \omega$ to the order type the initial segment of R_{α_n} below m. It is easy to see h is well-defined. Remember that any initial segment of any well-ordering is a well-ordering.

We show that $\operatorname{ran}(h) = \omega_1$. $\operatorname{ran}(h) \subset \omega_1$ is given by definition. We only need to check $\omega_1 \subset \operatorname{ran}(h)$. Take any $\alpha \in \omega_1$. Then there is a α_j for some $j \in \omega$ such that $\alpha \in \alpha_j$ by our assumption that $lim_{n\in\omega}\alpha_n = \omega_1$. Then $\{(k,l) : k,l \in \omega \text{ and } (k,l),(l,g_{\alpha_j}^{-1}(\alpha)) \in R_{\alpha_j}\}$ is isomorphic to α via g_{α_j} because $(m,n) \in R_{\alpha_j}$ if and only if $g_{\alpha_j}(m) \in g_{\alpha_j}(n) \in \alpha_j$, and hence

$$h(g_{\alpha_j}^{-1}(\alpha), j) = \alpha. \tag{2}$$

Thus we have shown $\operatorname{ran}(h) = \omega_1$. By the fact that $\omega \times \omega$ is countable, we get that ω_1 is countable. Contradiction. □

Steve Jackson gave a beautiful analysis of the regular cardinals below $\aleph_{\epsilon_0}$ in his PhD thesis [5]. We will not go into detail here, as for our present purposes, it only matters that there are uncountably many regular cardinals below Θ.

Theorem 5. *Assume* **AD**. *Then there are uncountably many regular cardinals below* Θ. *In particular,* ω_2 *is a regular cardinal.*

Proof. [5]; [6, p. 388]. □

Among regular cardinals below Θ, some are measurable. In particular, ω_1 and ω_2 are measurable [6, Theorem 28.2, Theorem 28.6].

4.1. **Real-coding games.** In this section we will see the interesting fact that our determinacy assumptions for games on ω implies certain games on higher cardinals are non-determined. Those non-determined games have similar style and we call them real-coding games.

If $S \subset \kappa < \Theta$ and π is a surjection from ${}^{\omega}\omega$ onto κ, we consider π as a **coding function** coding elements of κ by reals, amd define the set of π-codes for β to be

$$C^{\pi}_{\beta} = \{a \in \omega^{\omega} : \pi(a) = \beta\}.$$

Each $a \in C^{\pi}_{\beta}$ is called a π-code of β. For $S \subset \kappa$, we define the real-coding game for S and π as follows.

Definition 6. *Let* $S \subset \kappa < \Theta$, $\pi : {}^{\omega}\omega \to \kappa$ *be onto.*

The game G^{π}_{S} *is the strict game such that*

$$G^{\pi}_{S}(a) = \begin{cases} 0 & \text{if } a(0) \notin S \text{ or } \pi(a|_1) = a(0) \\ 1 & \text{otherwise} \end{cases}$$

for each $a \in {}^{\omega}\kappa$.

It is easy to see that the only move of O that matters is her first one and her first move α should be an element of S if she does not want to lose immediately. It is also easy to see P has to play some $b \in {}^{\omega}\omega$ if he wants to win at all. When the game is finished, P wins if and only $\alpha \notin S$ or $b \in C^{\pi}_{\alpha}$.

We give some interesting facts about real-coding games in the following.

Let $\omega < \kappa < \Theta$ and $\pi : {}^{\omega}\omega \to \kappa$ be onto. We define a function

$$\mathrm{FC}_{\pi} : \kappa \to P(\mathrm{Fin}(\omega))$$

that will give finite initial segments of π codes of β for each $\beta < \kappa$ as

$$\mathrm{FC}_{\pi}(\beta) = \{a{\restriction}n : n < \omega \ \wedge \ a \in C^{\pi}_{\beta}\}$$

for each $\beta < \kappa$.

It is easy to see $\mathrm{FC}_{\pi}(\beta)$ is the set of finite initial segments of π-codes of β, or finite codes of β in short.

If $R \subset \mathrm{Fin}(\omega)$ is such that for κ many α's, we have $\mathrm{FC}_{\pi}(\alpha) = R$, we say that R is FC_{π}-maximal. The following lemma states that FC_{π}-maximal sets exist.

Lemma 7. *There is $S \subset \kappa$ such that $|S| = \kappa$ and*

$$(\forall \alpha, \beta \in S)\ \mathrm{FC}_\pi(\alpha) = \mathrm{FC}_\pi(\beta).$$

Proof. It is easy to see $P(\mathrm{Fin}(\omega))$ has cardinality continuum. We already know that there is no uncountable well-orderable subsets of ${}^\omega\omega$, so the range of FC_π must be countable. By regularity of κ, there is a some $R \in \mathrm{ran}(\mathrm{FC}_\pi)$ such that $\{\beta : \mathrm{FC}_\pi(\beta) = R\}$ has cardinality κ. □

The following lemma says each FC_π-maximal set has size of the continuum.

Lemma 8. *If R is FC_π-maximal, then R has size continuum.*

Proof. Let $S \subset \kappa$ has cardinality κ and such that $\mathrm{FC}_\pi[S] = \{R\}$. Suppose R does not have size continuum. Then R is countable. $P = \{a : (\forall n < \omega)\ a{\restriction}n \in R\}$ is also countable. But $\pi[P] \supset S$, which has cardinality κ. Contradiction. □

The next lemma shows that for each $\beta \in S$, there are as many π codes of β as there are reals.

Lemma 9. *Let $S \subset \kappa$ has cardinality κ and such that $\mathrm{FC}_\pi[S] = \{R\}$. For each $\beta \in S$, C^π_β has cardinality continuum.*

Proof. Consider the function

$$(a, n) \mapsto a{\restriction}n.$$

This function is a surjection from $C^\pi_\beta \times \omega$ to R. If C^π_β is countable, then clearly $|R| = \omega$, contradicting our assumption. So C^π_β must have cardinality continuum. □

5. DEGREES OF REAL CODING GAMES

The reason that we are interested in these real-coding games is that they are non-determined. Let o be the infinite sequence of 0's, i.e., $o : \omega \to 1$. We will need this o several times in the rest of this thesis.

Theorem 10. *Assume* **AD**. *If $\omega < |\kappa| < \Theta$, π is a surjection from ${}^\omega\omega$ onto κ, $S \subset \kappa$ and $|S| > \omega$, then G^π_S is non-determined.*

Proof. Clearly O does not have a winning strategy, because for each $\alpha \in S$, there is some $a \in {}^\omega\omega$ such that $\pi(a) = \alpha$.

Suppose P has a winning strategy σ.

Define the function

$$f_\sigma(\alpha) = (\alpha * o) \star \sigma|_1.$$

Note that $(\alpha * o) \star \sigma|_1$ is the unique t such that P follows σ in the play $(\alpha * o) \star_0 t$. This is a choice function for the family $\{C^\pi_\beta : \beta \in S\}$ because σ is winning strategy for P. Since S is uncountable, this is a contradiction. □

Corollary 11 (Mycielski [7])**.** **AD** *implies there is a non-determined game on* ω_1*.*

Proof. Fix a 1-to-1 and onto function $f : {}^{\omega}\omega \to P(\omega \times \omega)$. Then $\pi_0 \circ f$ is a surjection from ${}^{\omega}\omega$ onto ω_1. □

Corollary 12. *Assume* **AD**. *If* $\omega < |\lambda| < \Theta$ *and* $\omega < |\kappa| < \Theta$*,* π *is a surjection from* ${}^{\omega}\omega$ *onto* λ *and* ρ *is a surjection from* ${}^{\omega}\omega$ *onto* κ*,* $S \subset \lambda$*,* $|S| > \omega$*,* $S' \subset \kappa$ *and* $|S'| > \omega$*, then player* O *does not have a winning strategy in the game* $G^{\pi}_{S} \mathbin{\widetilde{\otimes}} \widetilde{G^{\rho}_{S'}}$*.*

Proof. Consider the plays in which player P never makes switches. □

Another corollary is that **ZF** proves that all $\mathbf{AD}_{\omega_\alpha}$'s are false for $\alpha > 0$, since $\mathbf{AD} \to \neg\mathbf{AD}_{\omega_\alpha}$ and $\neg\mathbf{AD} \to \neg\mathbf{AD}_{\omega_\alpha}$.

5.1. **Partial Incomparability.** Corollary 12 tells us player O has no winning strategy in the game $G^{\pi}_{S} \mathbin{\widetilde{\otimes}} \widetilde{G^{\rho}_{S'}}$. What about player P? The following theorem says for some game of the form $G^{\pi}_{S} \mathbin{\widetilde{\otimes}} \widetilde{G^{\rho}_{S'}}$, P does not have a winning strategy.

Remember that if $\kappa > \lambda$, $G^{\rho}_{\kappa} \mathbin{\widetilde{\otimes}} \widetilde{G^{\pi}_{\lambda}}$ is $G^{\rho}_{\kappa} \mathbin{\widetilde{\otimes}} \mathrm{Ext}(\widetilde{G^{\pi}_{\lambda}}, \kappa)$.

Theorem 13. *Assume* **AD**. *Given a cardinal number* $\lambda > \omega$ *and a cardinal number* $\kappa > \lambda$*, a surjection* $\pi : {}^{\omega}\omega \to \lambda$ *and a surjection* $\rho : {}^{\omega}\omega \to \kappa$*, the game* $G^{\rho}_{S} \mathbin{\widetilde{\otimes}} \widetilde{G^{\pi}_{S'}}$ *is non-determined if* $|S|$ *is regular and* $|S| > |S'|$*. In particular,* $G^{\pi}_{S'} \not\leq G^{\rho}_{S}$*.*

Proof. Without loss of generality, we prove the case in which $S = \kappa$ and $S' = \lambda$.

By Corollary 12 O does not have a winning strategy. Now let us prove P does not have a winning strategy. Suppose, towards a contradiction, that σ is a winning strategy for P.

Notice that by definition of $\widetilde{\otimes}$, the first move in the game $G^{\rho}_{\kappa} \mathbin{\widetilde{\otimes}} \widetilde{G^{\pi}_{\lambda}}$ belongs to O and she has to play an ordinal in $\widetilde{G^{\pi}_{\lambda}}$. Now consider the set of all finished plays of $G^{\rho}_{\kappa} \mathbin{\widetilde{\otimes}} \widetilde{G^{\pi}_{\lambda}}$ in which P follows σ and O plays 0's in G^{ρ}_{κ} after her first move, and in which the the sub-game $\widetilde{G^{\pi}_{\lambda}}$ is unfinished. Let $\mathcal{P}_\sigma$ be the set of such plays. Formally

$$\mathcal{P}_\sigma = \{x \in {}^{\omega}\kappa : x = b \star \sigma \text{ for some } b \in {}^{\omega}\kappa \ \wedge\ (x)_{\widetilde{G^{\pi}_{\lambda}}} \in \mathrm{Fin}(\kappa)$$

$$\wedge\ (x)_{G^{\rho}_{\kappa}}|0 - (x)_{G^{\rho}_{\kappa}}(0) = o\}.$$

Clearly $|\mathcal{P}| = \kappa$. Define

$$\begin{aligned} S_\sigma &= \{\beta \in \kappa : O \text{ plays } \beta \text{ on her first move in } G^{\rho}_{\kappa} \text{ in some } x \in \mathcal{P}\} \\ &= \{\beta \in \kappa : (x)_{G^{\rho}_{\kappa}}(0) = \beta \text{ for some } x \in \mathcal{P}_\sigma\}. \end{aligned}$$

Lemma 14. S_σ *is at most countable.*

Proof of Lemma 21. Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays w's of $\widetilde{G^\pi_\lambda}$ (meaning $\mathrm{Ext}(\widetilde{G^\pi_\lambda}, \kappa)$). For each such w and $\beta \in S_\sigma$, there is at most 1 play x such that $x \in \mathcal{P}_\sigma$ and w is the $\widetilde{G^\pi_\lambda}$ part of x and O plays β in the sub-game G^ρ_κ. Formally,

$$g : (w \in \mathrm{Fin}(\lambda), \beta \in S_\sigma) \mapsto x \in \mathcal{P}_\sigma \text{ such that and } (x)_{\widetilde{G^\pi_\lambda}} = w \text{ and } x_{G^\rho_\kappa}(0) = \beta.$$

is a partial 1-to-1 function from $\mathrm{Fin}(\kappa) \times S_\sigma$ to $\mathcal{P}_\sigma$. (Take two such plays x_1 and x_2. The moves of player O are the same in both x_1 and x_2 and so must be the moves of P since P follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_\sigma$, define $T_\beta = \{(x)_{G^\rho_\kappa}|_1 : x \in \mathcal{P}_\sigma \ \wedge \ x_{G^\rho_\kappa}(0) = \beta\}$. T_β is a set of reals that code β, each of which is played by P in G^ρ_κ when $\widetilde{G^\pi_\lambda}$ is unfinished. Since g is 1-to-1, we know $1 \le |T_\beta| \le \kappa$.

By our assumption that S is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_\sigma} T_\beta$. Contradiction. □

Now consider all the plays of $G^\rho_\kappa \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_\lambda}$ in which O plays some $\beta \in \kappa - S_\sigma$ and 0's in G^ρ_κ. In each such play, $\widetilde{G^\pi_\lambda}$ is finished. And it is not hard to see that the infinite sequence played by O in $\widetilde{G^\pi_\lambda}$ could be anything. Since σ is a winning strategy for P, whenever O has played a proper code in $\widetilde{G^\pi_\lambda}$, P must have played a proper code in G^ρ_κ.

We need to define the following auxiliary objects from the winning strategy σ: Given $\beta \in \kappa - S_\sigma$, let $r_\sigma(\beta)$ be the move in $\widetilde{G^\pi_\lambda}$ that player P makes according to σ in the game G^π_λ after player O played β in G^ρ_κ.

After β and $r_\sigma(\beta)$ have been played, the winning strategy sigma gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for β, a continuous function that maps each $a \in C^\pi_{r_\sigma(\beta)}$ to some $b \in C^\rho_\beta$.

To make it precise, let $F^\beta_\sigma = \{s : s \text{ is a finite play of } G^\rho_\kappa \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_\lambda}, P \text{ follows } \sigma \text{ in } s, (s)_{G^\rho_\kappa}|_0 \prec \beta * o \text{ and } (s)_{\widetilde{G^\pi_\lambda}}|_0 \in \mathrm{Fin}(\omega)\}$ and $f^\beta_\sigma = \{(t,p) : (\exists s \in F^\beta_\sigma)\ t = (s)_{G^\pi_\lambda}|_0 \ \wedge \ p = (s)_{\widetilde{G^\rho_\kappa}}|_1\}$. Clearly $f^\beta_\sigma \subset \mathrm{Fin}(\omega) \times \mathrm{Fin}(\omega)$ and f^β_σ is a function since σ is a strategy for P. Let $\mathcal{F}_\sigma = \{f^\beta_\sigma : \beta \in \kappa - S_\sigma\}$. By Theorem **??**,

$$|\mathcal{F}_\sigma| \le |P(\omega \times \omega)| = |\omega^\omega|. \tag{3}$$

From f^β_σ we get very naturally a continuous function $\hat{f}^\beta_\sigma : C^\pi_{r_\sigma(\beta)} \to C^\rho_\beta$ defined by

$$\hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a{\restriction}n). \tag{4}$$

Because by the definition of f_σ^β and the fact that σ is a winning strategy for P in $G_\kappa^\rho \,\widetilde{\otimes}\, \widetilde{G_\lambda^\pi}$, for any given $a \in C^\pi_{r_\sigma(\beta)}$, i.e., $\pi(a) = r_\sigma(\beta)$,

$$\rho(\bigcup_{n<\omega} f_\sigma^\beta(a\restriction n)) = \beta$$

which is

$$\hat{f}_\sigma^\beta(a) = \bigcup_{n<\omega} f_\sigma^\beta(a\restriction n) \in C_\beta^\rho. \tag{5}$$

Consider the function $r_\sigma : \kappa - S_\sigma \to \lambda$. Because $|\kappa - S_\sigma| = \kappa$ and κ is regular, there must be some α' in $r_\sigma(\kappa - S_\sigma)$ such that $X := \{\beta : r_\sigma(\beta) = \alpha'\}$ has cardinality κ. Take $\beta_0 \neq \beta_1$ in X and consider $f_\sigma^{\beta_0}$ and $f_\sigma^{\beta_1}$. Suppose $\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'$. Then $\rho(\hat{f}_\sigma^{\beta_0}(x)) = \beta_0$ and $\rho(\hat{f}_\sigma^{\beta_1}(x)) = \beta_1$, which implies $\hat{f}_\sigma^{\beta_0} \neq \hat{f}_\sigma^{\beta_1}$ and hence $f_\sigma^{\beta_0} \neq f_\sigma^{\beta_1}$. Thus, the set $\mathcal{F}_\sigma^X = \{f_\sigma^\beta : \beta \in X\}$ has cardinality κ. By $\mathcal{F}_\sigma^X \subset \mathcal{F}_\sigma$ and 3, we get a subset of ω^ω of size κ which contradicts AD. □

Corollary 15. *Assume* **AD***. Given a surjection π from ${}^\omega\omega$ onto ω_1 and a surjection ρ from ${}^\omega\omega$ onto ω_2, $\widetilde{G_{\omega_1}^\pi} \,\widetilde{\otimes}\, G_{\omega_2}^\rho$ is non-determined. In particular, $G_{\omega_1}^\pi \not\leq G_{\omega_2}^\rho$.*

Corollary 16. *Assume* **AD***. Given an infinite cardinal number $\kappa > \omega$ and, a surjection $\pi : {}^\omega\omega \to \kappa$, if $|S|$ is regular and $|S| > |S'|$, the game $G_S^\pi \,\widetilde{\otimes}\, \widetilde{G_{S'}^\pi}$ is non-determined. In particular, $G_{S'}^\pi \not\leq G_S^\pi$.*

Proof. Trivial modification of the proof of Theorem 13. □

Corollary 17. *If $\kappa > \omega$ is regular under* **AD***, $\mathcal{S}_\kappa$ has at least four different degrees.*

It would be nice if we have also shown in Theorem 13 that $G_S^\rho \not\leq G_{S'}^\pi$, because that would imply $\mathcal{S}_\kappa$ has more than one degrees. However, we can not show that in general. Under **AD**, we can construct games of the form $G_\lambda^\pi \,\widetilde{\otimes}\, \widetilde{G_\kappa^\rho}$ with $\lambda < \kappa$ in which P has a winning strategy.

Given a cardinal number $\lambda > \omega$ and a cardinal number $\kappa > \lambda$, a surjection π from ${}^\omega\omega$ onto λ and a surjection ρ from ${}^\omega\omega$ onto κ. Define $T_\lambda^\kappa = \{\alpha \in \kappa : (\forall\beta < \alpha)(\forall\gamma < \lambda)\ \beta + \gamma \neq \alpha\}$. $|T_\lambda^\kappa| = \kappa$ since $\lambda^\gamma \in T_\lambda^\kappa$ for all $\gamma < \kappa$. From ρ and the order-type function from T_λ^κ to κ, we can get a surjection $\psi : {}^\omega\omega \to T_\lambda^\kappa$. Define $\chi : {}^\omega\omega \to \kappa$ by

$$\chi(a) = \psi(a_0) + \pi(a_1)$$

where a_0 is the even part of a and a_1 is the odd part of a. It is easy to see that the range of χ is κ.

P has the following winning strategy in $G_\lambda^\pi \,\widetilde{\otimes}\, \widetilde{G_\kappa^\chi}$: After O plays α in G_λ^π, switch to $\widetilde{G_\kappa^\chi}$ and play $\lambda + \alpha$. Keep switching between two games while copying the odd part of O's play in $\widetilde{G_\kappa^\chi}$ into your (P's) own code played in G_λ^π.

We apply the partition property of ω_1 to certain family of ω_1 many games to get results about comparability among these games.

Lemma 18. *Let $\pi : {}^{\omega}\omega \to \omega_1$ be onto. Let $\{S_\alpha\}_{\alpha<\omega_1}$ be a partition of ω_1 such that $|S_\alpha| = \omega_1$ for all $\alpha < \omega_1$. There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $G^\pi_{S_\alpha} \| G^\pi_{S_\beta}$ for all $\alpha \in \beta \in I$ or 2. $G^\pi_{S_\alpha} \leq G^\pi_{S_\beta}$ or $G^\pi_{S_\beta} \leq G^\pi_{S_\alpha}$ for all $\alpha \in \beta \in I$.*

Proof. Easy application of $\omega_1 \to (\omega_1)^2_2$. □

The same argument can be applied to the first ω_1 regular cardinals. Let κ_α be the α-th uncountable regular cardinal under **AD**. By Theorem 5 these are all less than Θ, so there is a surjections ρ_α from ${}^{\omega}\omega$ onto each κ_α. Moreover, these ρ_α's can be precisely defined without using any choice; see [6, p. 397–398]. Let H_α be the game $G^{\rho_\alpha}_{\kappa_\alpha}$. Let $\mathcal{H}$ be the set of all these H_α's.

Theorem 19. *There is a set $I \subset \omega_1$ such that $|I| = \omega_1$ and either 1. $H_\alpha \| H_\beta$ for all $\alpha \in \beta \in I$ or 2. $H_\alpha > H_\beta$ for all $\alpha \in \beta \in I$. That is, there is an uncountable strictly increasing sequence of games in $\mathcal{H}$, or there is an uncountable set of incomparable games in $\mathcal{H}$.*

Proof. For any $\alpha < \beta$, $H_\alpha \not\leq H_\beta$ by Theorem 13 and hence either $H_\alpha > H_\beta$ or $H_\alpha \| H_\beta$.

Define the following colouring function:

$$c(\{\alpha,\beta\}) = \begin{cases} 1 & \text{if } \alpha < \beta \to H_\alpha > H_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

The partition property gives us a homogeneous set T for c of cardinality ω_1, and this is either an uncountable strictly increasing sequence of games (if it is homogeneous for 1) or an uncountable set of incomparable games (if it is homogeneous for 0). □

Corollary 20. *The degree universe $\mathcal{S}_\Theta$ has chains of order-type ω_1 or antichains of cardinality ω_1.*

6. The NEUCL assumption

Consider the following question: given a surjection $\pi : {}^{\omega}\omega \to \kappa$ where κ is an infinite regular cardinal $> \omega$, what S and S' satisfy that $|S'| = |S| = \kappa$ and P has no winning strategy in $G^\pi_S \mathbin{\tilde{\otimes}} \widetilde{G^\pi_{S'}}$?

Let $\kappa > \omega$ be a regular cardinal. Consider the game $G^\pi_{S_0} \mathbin{\tilde{\otimes}} \widetilde{G^\pi_{S_1}}$ where $S_0, S_1 \subset \kappa$ and $|S_0| = |S_1| = \kappa$. By Corollary 12 O does not have a winning strategy. Now let us suppose that σ is a winning strategy for P. We do the same analysis as in the proof of Theorem 13.

Notice that by definition of $\tilde{\otimes}$, the first move in the game $G^\pi_{S_0} \mathbin{\tilde{\otimes}} \widetilde{G^\pi_{S_1}}$ belongs to O and she has to play an ordinal in $\widetilde{G^\pi_{S_1}}$. Now consider the set of all finished plays of $G^\pi_{S_0} \mathbin{\tilde{\otimes}} \widetilde{G^\pi_{S_1}}$ in which P follows σ and O plays 0's in

$G^\pi_{S_0}$ after her first move, and in which the the sub-game $\widetilde{G^\pi_{S_1}}$ is unfinished. Let $\mathcal{P}_\sigma$ be the set of such plays. Formally

$$\mathcal{P}_\sigma = \{x \in {}^\omega\kappa : x = b \star \sigma \text{ for some } b \in {}^\omega\kappa \ \wedge \ (x)_{\widetilde{G^\pi_{S_1}}} \in \mathrm{Fin}(\kappa)$$
$$\wedge \ (x)_{G^\pi_{S_0}}|0 - (x)_{G^\pi_{S_0}}(0) = o\}.$$

Clearly $|\mathcal{P}| = \kappa$. Define

$$\begin{aligned} S_\sigma &= \{\beta \in S_0 : O \text{ plays } \beta \text{ on her first move in } G^\pi_{S_0} \text{ in some } x \in \mathcal{P}\} \\ &= \{\beta \in S_0 : (x)_{G^\pi_{S_0}}(0) = \beta \text{ for some } x \in \mathcal{P}_\sigma\}. \end{aligned}$$

Lemma 21. *The set S_σ is at most countable.*

Proof of Lemma 21. Suppose not.

There are at most $\kappa \times \omega = \kappa$ many finite plays w's of $\widetilde{G^\pi_{S_1}}$ (meaning $\mathrm{Ext}(\widetilde{G^\pi_{S_1}}, \kappa)$). For each such w and $\beta \in S_\sigma$, there is at most 1 play x such that $x \in \mathcal{P}_\sigma$, w is the $\widetilde{G^\pi_{S_1}}$ part of x and O plays β in the sub-game $G^\pi_{S_0}$. Formally,

$$g : (w \in \mathrm{Fin}(\lambda), \beta \in S_\sigma) \mapsto x \in \mathcal{P}_\sigma \text{ such that and } (x)_{\widetilde{G^\pi_{S_1}}} = w \text{ and } x_{G^\pi_{S_0}}(0) = \beta.$$

is a partial 1-to-1 function from $\mathrm{Fin}(\kappa) \times S_\sigma$ to $\mathcal{P}_\sigma$. (Take two such plays x_1 and x_2. The moves of player O are the same in both x_1 and x_2 and so must be the moves of P since P follows a strategy, which implies $x_1 = x_2$.)

For each $\beta \in S_\sigma$, define $T_\beta = \{(x)_{G^\pi_{S_0}}|_1 : x \in \mathcal{P}_\sigma \ \wedge \ x_{G^\pi_{S_0}}(0) = \beta\}$. T_β is a set of reals that code β, each of which is played by P in $G^\pi_{S_0}$ when $\widetilde{G^\pi_{S_1}}$ is unfinished. Since g is 1-to-1, we know $1 \leq |T_\beta| \leq \kappa$.

By our assumption that S is uncountable, we get a well-orderable set of reals $\bigcup_{\beta \in S_\sigma} T_\beta$. Contradiction. □

Now consider all the plays of $G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}$ in which O plays some $\beta \in S_0 - S_\sigma$ and 0's in $G^\pi_{S_0}$. In each such play, $\widetilde{G^\pi_{S_1}}$ is finished. And it is not hard to see that the infinite sequence played by O in $\widetilde{G^\pi_{S_1}}$ could be anything. Since σ is a winning strategy for P, whenever O has played a proper code in $\widetilde{G^\pi_{S_1}}$, P must have played a proper code in $G^\pi_{S_0}$.

We need to define the following auxiliary objects from the winning strategy σ: Given $\beta \in S_0 - S_\sigma$, let $r_\sigma(\beta)$ be the move in $\widetilde{G^\pi_{S_1}}$ that player P makes according to σ in the game $G^\pi_{S_1}$ after player O played β in $G^\pi_{S_0}$.

After β and $r_\sigma(\beta)$ have been played, the winning strategy sigma gives a definition of a continuous function reducing a code for $r_\sigma(\beta)$ into a code for β, a continuous function that maps each $a \in C^\pi_{r_\sigma(\beta)}$ to some $b \in C^\pi_\beta$.

To make it precise, let $F^\beta_\sigma = \{s : s \text{ is a finite play of } G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}, P \text{ follows } \sigma \text{ in } s, (s)_{G^\pi_{S_0}}|_0 \prec \beta * o \text{ and } (s)_{\widetilde{G^\pi_{S_1}}}|_0 \in \mathrm{Fin}(\omega)\}$ and $f^\beta_\sigma = \{(t, p) : (\exists s \in$

F^β_σ) $t = (s)_{G^\pi_{S_1}}|_0 \ \wedge \ p = (s)_{\widetilde{G^\pi_{S_0}}}|_1\}$. Clearly $f^\beta_\sigma \subset \mathrm{Fin}(\omega) \times \mathrm{Fin}(\omega)$ and f^β_σ is a function since σ is a strategy for P. Let $\mathcal{F}_\sigma = \{f^\beta_\sigma : \beta \in S_0 - S_\sigma\}$. As before,

$$|\mathcal{F}_\sigma| \leq |P(\omega \times \omega)| = |\omega^\omega|. \tag{6}$$

From f^β_σ we get very naturally a continuous function $\hat{f}^\beta_\sigma : C^\pi_{r_\sigma(\beta)} \to C^\pi_\beta$ defined by

$$\hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a \restriction n). \tag{7}$$

Because by the definition of f^β_σ and the fact that σ is a winning strategy for P in $G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}$, for any given $a \in C^\pi_{r_\sigma(\beta)}$, i.e., $\pi(a) = r_\sigma(\beta)$,

$$\pi(\bigcup_{n<\omega} f^\beta_\sigma(a \restriction n)) = \beta$$

which is

$$\hat{f}^\beta_\sigma(a) = \bigcup_{n<\omega} f^\beta_\sigma(a \restriction n) \in C^\pi_\beta. \tag{8}$$

Consider the function $r_\sigma : S_0 - S_\sigma \to S_1$.

Lemma 22. $|\mathrm{ran}(r_\sigma)| = \kappa$.

Proof. Because $|S_0 - S_\sigma| = \kappa$ and κ is regular, there must be some $\alpha' \in \mathrm{ran}(r_\sigma)$ such that $X := \{\beta : r_\sigma(\beta) = \alpha'\}$ has cardinality κ. Take $\beta_0 \neq \beta_1$ in X and consider $f^{\beta_0}_\sigma$ and $f^{\beta_1}_\sigma$. Suppose $\pi(x) = r_\sigma(\beta_0) = r_\sigma(\beta_1) = \alpha'$. Then $\pi(\hat{f}^{\beta_0}_\sigma(x)) = \beta_0$ and $\pi(\hat{f}^{\beta_1}_\sigma(x)) = \beta_1$, which implies $\hat{f}^{\beta_0}_\sigma \neq \hat{f}^{\beta_1}_\sigma$ and hence $f^{\beta_0}_\sigma \neq f^{\beta_1}_\sigma$. Thus, the set $\mathcal{F}_\sigma = \{f^\beta_\sigma : \beta \in X\}$ has cardinality κ. By $\mathcal{F}_\sigma \subset \mathcal{F}_\sigma$ and 6, we get a subset of ω^ω of size κ which contradicts AD. □

Define $g_\sigma : \mathrm{ran}(r_\sigma) \to S_0 - S_\sigma$ by

$$g_\sigma(\alpha) = \text{ the least } \beta \in S_0 - S \text{ such that } r_\sigma(\beta) = \alpha.$$

Clearly g_σ is 1-to-1 and has range of cardinality κ. It is easy to see r_σ is 1-to-1 on $\mathrm{ran}(g_\sigma)$. Since $\mathcal{F}_\sigma$ is countable and κ is regular, we know there is $S^*_0 \subset \mathrm{ran}(g_\sigma)$ and $f \in F_\sigma$ such that $|S^*_0| = \kappa$ and $(\forall \beta, \beta' \in S^*_0)\ f^\beta_\sigma = f^{\beta'}_\sigma = f$.

Thus we get a continuous function $\hat{f} : \bigcup_{\beta \in S^*_0} C^\pi_{r_\sigma(\beta)} \to \bigcup_{\beta \in S^*_0} C^\pi_\beta$. $\hat{f}$ is a universal reduction function in the sense that $\hat{f} \restriction C^\pi_{r_\sigma(\beta)} = \hat{f}^\beta_\sigma$, or equivalently

$$(\forall \beta \in S^*_0)\ \hat{f}[C^\pi_{r_\sigma(\beta)}] \subset C^\pi_\beta.$$

Definition 23. *Let* $\pi : {}^\omega\omega \to \kappa$ *be a surjection,* $S \subset \kappa$ *and* $g : S \to \kappa$ *be 1-to-1. A continuous function* f *that is (partially) defined on* ${}^\omega\omega$ *is a* **π-universal continuous lifting** *for* S *and* g *iff*

$$(\forall \beta \in S) f[C^\pi_\beta] \subset C^\pi_{g(\beta)}.$$

It is easy to see the above $\hat{f}$ is π-universal for $r_\sigma[S_0^*]$ and g_σ.

From the above discussion, we know that if such a universal continuous lifting cannot exists, then we have two games that are incomparable. We now prove this more precisely.

Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. We say the **NEUCL Assumption** (NEUCL for "non-existence of universal continuous lifting") holds for the pair (κ, π) if and only if there is no π-universal continuous lifting f for any $S \subset \kappa$ with $|S| = \kappa$ and any 1-to-1 $g : S \to \kappa$ such that $g \cap \mathrm{id} = \emptyset$.

Lemma 24. *Assume* **AD**. *Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for (κ, π), then for any S_0 and S_1 such that $|S_0| = |S_1| = \kappa$ and $|S_0 \cap S_1| < \kappa$, it is true that $G^\pi_{S_0} \| G^\pi_{S_1}$.*

Proof. By the above analysis, P does not have a winning strategy in either $G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}$ or $G^\pi_{S_1} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_0}}$; by Corollary 12, O does not have a winning strategy in either $G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}$ or $G^\pi_{S_1} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_0}}$. □

It is not hard to see that if the premiss of Lemma 24 holds, we get a large family of pairwise incomparable games. The following theorem says that if the NEUCL Assumption holds for some (κ, π), then $\mathcal{S}_\kappa$ has an antichain of size κ.

Theorem 25. *Assume* **AD**. *Let $\kappa > \omega$ be a regular cardinal and $\pi : {}^\omega\omega \to \kappa$ be a surjection. If the NEUCL Assumption holds for (κ, π), then there is a family of games on κ $\{G_\alpha : \alpha \in \kappa\}$ such that $G_\alpha \| G_\beta$ for any $\alpha \neq \beta$.*

Proof. Use the canonical partition of κ and get $S_\alpha = \kappa|\alpha$ for each $\alpha \in \kappa$. Apply Lemma 24 to S_α and S_β for any $\alpha \neq \beta$. □

Corollary 26. *Let $\kappa > \omega$ be a regular cardinal under* **AD**. *If there is a onto function $\pi : {}^\omega\omega \to \kappa$ such that the NEUCL Assumption holds for (κ, π), then $\mathcal{S}_\kappa$ has antichains of size κ.*

The NEUCL Assumption cannot hold for each pair (κ, π). The following is a counterexample.

Recall that $\Gamma \restriction \omega \times \omega : \omega \times \omega \to \omega$ is 1-to-1 and onto. Define $\mathcal{R} : {}^\omega 2 \to P(\omega \times \omega)$ by

$$(m, n) \in \mathsf{R}(a) \text{ iff } f(\Gamma(m, n)) = 1$$

for each $a \in {}^\omega 2$. Clearly $\mathcal{R}$ is 1-to-1 and onto. A nice property of $\mathcal{R}$ that will be useful is that $\mathsf{R}(a) \restriction n \times n$ can be read from the first $\Gamma(0, n)$ bits of a. It is so by our choice of Γ, which was defined earlier.

Define $\pi : {}^\omega\omega \to \omega_1$ by

$$\pi(a) = \begin{cases} \pi_0 \circ \mathcal{R}(a) & \text{if } a \in {}^\omega 2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that π_0 is the canonical projection : $P(\omega \times \omega) \to \omega_1$. So π is a well-defined projection : ${}^\omega\omega \to \omega_1$.

Let $S_1 = \{\omega^\alpha : \alpha < \omega_1\}$ and $S_0 = \{\omega^\alpha \cdot 2 : \alpha < \omega_1\}$. Clearly $|S_0| = |S_1| = \kappa$ and $S_0 \cap S_1 = \emptyset$. We show that P has the following winning strategy σ in $G^\pi_{S_0} \mathbin{\widetilde{\otimes}} \widetilde{G^\pi_{S_1}}$.

By definition of $\widetilde{\otimes}$, the game starts with O play a $\alpha \in S_0$ in the sub-game $G^\pi_{S_0}$. After O's starting move, P play the unique β such that $\beta \cdot 2 = \alpha$ in the sub-game $\widetilde{G^\pi_{S_1}}$. Then O must begin to play a code of β, if she wants to win at all. Whenever O has played $\Gamma(0, n+1)$ bits of her code b, interrupt her and extend your (P's) code a in $G^\pi_{S_0}$ to the first $\Gamma(0, 2n)$ bits so that

1. for all $m, k < n$, if $b(\Gamma(m,k)) = 1$ then $a(\Gamma(2m, 2k)) = 1$ and $a(\Gamma(2m+1, 2k+1)) = 1$,
2. for all $m, m', k, k' < n$, if $b(\Gamma(m,k)) = 1$ and $b(\Gamma(m',k')) = 1$, then $(\forall x, y \in \{m, m', k, k'\})\ a(\Gamma(2x, 2y+1)) = 1$,
3. $a(k) = 0$ for all $k < 2n$ not covered by the above clauses.

After extending a to the first $2n$ bits, switch to $\widetilde{G^\pi_{S_1}}$ and let O continue to play her code b.

It is easy to see this is a well-defined strategy. Note that if O starts to play anything other than 0-1 bits at any point, you just need to sit and watch and you win the game when it is finished.

It is easy to see for each $n < \omega$, if $b \restriction n$ codes a well-ordering of order-type m, $a \restriction 2n$ codes a well-ordering of order-type $m \cdot 2$, and when the game is finished, if b codes a well-ordering of order-type $\alpha \in S_0$, a codes $\alpha \cdot 2$. So the strategy is a winning one for P.

From the definition of P's strategy σ, $\mathcal{F}^\beta_\sigma = \mathcal{F}^{\beta'}_\sigma$ and hence $f^\beta_\sigma = f^{\beta'}_\sigma = f_\sigma$ for each $\beta, \beta' \in S_0$. $\hat{f}_\sigma$ is π-universal for S_0 and $g : \beta \mapsto \beta \cdot 2$, i.e.,

$$(\forall \beta \in S_0)\ \hat{f}_\sigma[C^\pi_\beta] \subset C^\pi_{\beta \cdot 2}.$$

We have seen that the NEUCL Assumption cannot be a general theorem for all pair (κ, π) where κ is regular under **AD** and $\pi : {}^\omega\omega \to \kappa$ is onto. It is not easy to see what (κ, π) can validate the NEUCL Assumption. Now let us consider given (κ, π) what $S \subset \kappa$ and $g : S \to \kappa$ cannot falsify the NEUCL Assumption.

First, we consider the case in which $S \subset \kappa = \omega_1$ and $g : S \to S$ is 1-to-1 and onto, and

$$(\forall \alpha \in S)\ g(\alpha) \neq \alpha.$$

Lemma 27. *There is a set $T \subset S$ of cardinality κ such that $g[T] \cap T = \emptyset$*

Proof. It is well known that **AD** implies $\omega_1 \to (\omega_1)^2_2$.

Define a colouring function $c : [\omega]^2 \to 2$ by

$$c(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } g(\alpha) = \beta \text{ or } g(\beta) = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

By weakly compactness, there is a homogenous set $T \subset S$ of cardinality κ. We claim $c[T] = \{0\}$.

Suppose otherwise, $c[T] = \{1\}$. Let α, β, γ and δ are four different elements of T. Without loss of generality, we can assume that $g(\alpha) = \beta$, which implies $g(\alpha) \neq \gamma, \delta$. But since $c(\{\alpha, \gamma\}) = c(\{\alpha, \delta\}) = 1$, it has to be the case $g(\gamma) = g(\delta) = \alpha$. But that means that $g(\gamma) \neq \delta$ and $g(\delta) \neq \gamma$ and hence $c(\{\gamma, \delta\}) = 0$. Contradiction.

By the definition of c, $g(\alpha) \neq \beta$ for all $\alpha, \beta \in T$ and hence $g[T] \cap T = \emptyset$. □

Clearly if f is a π-universal continuous lifting for S and g, it is also a π-universal continuous lifting for T and g. So if we can show that for any $S \subset \omega_1$ and any g such that $|S| = \omega_1$ and $g[S] \cap S = \emptyset$, there is no π-universal continuous lifting, then we also have that for any $S \subset \omega_1$ and any 1-to-1 and onto $g : S \to S$ such that $(\forall \alpha \in S)\ g(\alpha) \neq \alpha$, there is no π-universal continuous lifting.

References

[1] Andreas Blass. Degrees of indeterminacy of games. *Fund. Math.*, 77(2):151–166, 1972.
[2] Andreas Blass. A game semantics for linear logic. *Ann. Pure Appl. Logic*, 56(1-3):183–220, 1992.
[3] G. Cantor. De la puissance des ensembles parfaits de points. *Acta Math.*, 4(1):381–392, 1884. Extrait d'une lettre adressée à l'éditeur.
[4] Georg Cantor. Ueber unendliche, lineare Punktmannichfaltigkeiten. *Math. Ann.*, 21(4):545–591, 1883.
[5] Stephen C. Jackson. *A calculation of* $\boldsymbol{\delta}^1_5$. PhD thesis, UCLA, 1983.
[6] Akihiro Kanamori. *The higher infinite*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003. Large cardinals in set theory from their beginnings.
[7] Jan Mycielski. On the axiom of determinateness. *Fund. Math.*, 53:205–224, 1963/1964.